Anomaly-Free Supersymmetric $\frac{SO(2N+2)}{U(N+1)}$ σ-Model Based on the $SO(2N+1)$ Lie Algebra of the Fermion Operators*

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Dedicated to the Memory of Hideo Fukutome

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Abstract

The extended supersymmetric (SUSY) σ-model has been proposed on the bases of $SO(2N+1)$ Lie algebra spanned by fermion annihilation-creation operators and pair operators. The canonical transformation, extension of an $SO(2N)$ Bogoliubov transformation to an $SO(2N+1)$ group, is introduced. Embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using $SO(2N+2)/U(N+1)$ coset variables, we have investigated the SUSY σ-model on the Kähler manifold, the coset space $SO(2N+2)/U(N+1)$. We have constructed the Killing potential, extension of the potential in the $SO(2N)/U(N)$ coset space to that in the $SO(2N+2)/U(N+1)$ coset space. It is equivalent to the generalized density matrix whose diagonal-block part is related to a reduced scalar potential with a Fayet-Ilipoulos term. The f-deformed reduced scalar potential is optimized with respect to vacuum expectation value of the σ-model fields and a solution for one of the $SO(2N+1)$ group parameters has been obtained. The solution, however, is only a small part of all solutions obtained from anomaly-free SUSY coset models. To construct the coset models consistently, we must embed a coset coordinate in an anomaly-free spinor representation (rep) of $SO(2N+2)$ group and give corresponding Kähler and Killing potentials for an anomaly-free $SO(2N+2)/U(N+1)$ model based on each positive chiral spinor rep. Using such mathematical manipulation we construct successfully the anomaly-free $SO(2N+2)/U(N+1)$ SUSY σ-model and investigate new aspects which have never been seen in the SUSY σ-model on the Kähler coset space $SO(2N)/U(N)$. We reach a f-deformed reduced scalar potential. It is minimized with respect to the

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vacuum expectation value of anomaly-free SUSY σ-model fields. Thus we find an interesting f-deformed solution very different from the previous solution for an anomaly-free $SO(2\cdot\mathbf{5}+2)/(SU(\mathbf{5}+1)\times U(1))$ SUSY σ-model.

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1 Introduction

The supersymmetric (SUSY) extension of the nonlinear σ-model was first given by Zumino under the introduction of scalar fields [1] which take values in a complex Kähler manifold. The extended σ-model defined on symmetric spaces have been intensively studied in modern elementary particle physics, superstring theory and supergravity theory [2]. The σ-model on the hyper Kähler manifold also has been deeply investigated in various contexts [3].

The Hartree-Bogoliubov theory (HBT) [4] has been regarded as the standard approximation in the theory of fermion systems [5]. In the HBT an HB wave function for such systems represents a Bose condensate of fermion pairs. Standing on the Lie-algebraic viewpoint, the fermion pair operators form an $SO(2N)$ Lie algebra and contain a $U(N)$ Lie algebra as a subalgebra where N denotes the number of fermion states. The $SO(2N)(=g)$ and $U(N)(=h)$ mean a special orthogonal group of $2N$ dimensions and a unitary group of N dimensions, respectively. One can give an integral representation of a state vector under the group g, the exact coherent state representation (CS rep) of a fermion system [6].

A procedure for consistent coupling of gauge- and matter superfields to SUSY σ-models on the Kähler coset spaces has been given by van Holten et al. These authors have presented a mathematical tool of constructing a Killing potential and have applied their method to the explcit construction of SUSY σ-models on the coset spaces $\frac{SO(2N)}{U(N)}$. They have shown that only a finite number of the coset models can be consistent when coupled to matter superfields with $U(N)$ quantum numbers reflecting spinor reps of $SO(2N)$ [2]. Deldug and Valent have investigated the Kählerian σ-models in two-dimensional space-time at the classical and quantum levels. They have presented a unified treatment of the models based on irreducible hermitian symmetric spaces corresponding to the coset spaces $\frac{G}{H}$ [7]. On the other hand, van Holten et al. and Higasijima et al. have also discussed the construction of σ-models on compact and non-compact Grassmannian manifolds, $\frac{SU(N+M)}{S[U(N)\times U(M)]}$ and $\frac{SU(N,M)}{S[U(N)\times U(M)]}$ [8, 9].

Fukutome et al. have proposed a new fermion many-body theory based on the $SO(2N{+}1)$ Lie algebra of fermion operators composed of single annihilation c_α and creation $c_\alpha^\dagger$ operator $(\alpha = 1, \ldots, N)$ and pair operator [10]. A rep of an $SO(2N{+}1)$ group has been derived by a group extension of the $SO(2N)$ Bogoliubov transformation for fermions to a new canonical transformation group. The fermion Lie operators, when operating on the integral rep of the $SO(2N{+}1)$ wave function, are mapped into the regular rep of the $SO(2N{+}1)$ group and are represented by boson operators. Bosonization of creation-annihilation and pair operators is given in [11].

Along the same strategy we have proposed an extended SUSY σ-model on Kähler coset space $\frac{G}{H} = \frac{SO(2N+2)}{U(N+1)}$ based on the $SO(2N{+}1)$ Lie algebra of the fermion operators [12] (referred to as I). Embedding the $SO(2N{+}1)$ group into an $SO(2N{+}2)$ group and using $\frac{SO(2N+2)}{U(N+1)}$ coset variables [13], we have studied a new aspect of the SUSY σ-model on the Kähler manifold of the coset space $\frac{SO(2N+2)}{U(N+1)}$. If we introduce annihilation and creation operators, c_{N+1} and $c_{N+1}^\dagger$, for a fictitious degree of freedom $N{+}1$, an unphysical space of 2^{N+1} dimensions is inevitably met. Then the $\frac{SO(2N+2)}{U(N+1)}$ coset description under the algebra $SO(2N{+}2)$ is essential and important. In this context we have constructed a Killing potential. It is greatly surprising that the Killing potential is equivalent to the generalized density matrix in the HBT. Its diagonal-block part is related to a reduced scalar potential with a Fayet-Ilipoulos term. After rescaling the Goldstone fields by a parameter f (inverse of mass m_σ), minimization of the f-deformed reduced scalar

potential has led us to an interesting solution of the SUSY σ-mdoel.

The solution of the $\frac{SO(2N+2)}{U(N+1)}$ SUSY σ-model, however, is only a small part of all solutions obtained from anomaly-free SUSY coset models. A consistent theory of coupling of gauge- and matter-superfields to SUSY σ-model has been proposed on the Kähler coset space. As shown by van Holten et al. [2], if we construct some quantum field theories based on pure coset models, we meet with a serious problem of anomalies in a holonomy group which particularly occur in pure SUSY coset models due to the existence of chiral fermions. Using a coset coordinate in an anomaly-free spinor rep of the $SO(2N)$ group, these authors have constructed a Killing potential and have applied their method to the explicit construction of the SUSY σ-model on the coset space $\frac{SO(2N)}{U(N)}$. For analysis of the anomaly see Ref. [14]. The anomaly cancellation condition was first given by Georgi and Glashow [15]. The Adler-Bell-Jackiw anomaly often occurs in gauged-SUSY nonlinear σ-models jointly with σ fermions [16, 17]. The σ-models are based on the Kähler manifolds $\frac{G}{H}$ (H: Subgroup of G). di Vecchia et al. stated that some nonlinear σ-models in which the scalar manifold is a coset space $\frac{G}{H}$ may show up anomalies when they are coupled to fermions which are not in anomaly-free reps of the subgroup H of G [18]. Anomalies in compact and non-compact Kähler manifolds have been intensively discussed [19, 20]. This is also the case for our orthogonal coset $\frac{SO(2N+2)}{U(N+1)}$, though a spinor rep of the $SO(2N+2)$ group is anomaly free. To construct a consistent SUSY coset model, we must embed a coset coordinate in an anomaly-free spinor rep of $SO(2N+2)$ group and give corresponding Kähler and Killing potential for the anomaly-free $\frac{SO(2N+2)}{U(N+1)}$ model based on a positive chiral spinor rep. To achieve such an object in the case of $SO(2N)$ group/algebra, van Holten et al. have proposed a method of constructing the Kähler and Killing potentials [2]. This idea is very suggestive and useful for our present aim of constructing the corresponding Kähler and Killing potentials for the case of $SO(2N+2)$ group/algebra.

The $\frac{SO(10)}{U(5)}$ coset model is proposed in the Standard Model and constructed by the $SU(5)\times U(1)$ fermionic fields content of one generation of quarks and leptons, including a right-handed neutrino. There is, however, no coset model such as $\frac{SO(12)}{U(6)}$ in the Standard Model. Thus we choose the $\frac{SO(10)}{U(5)}$ coset model as a basic model and extend it to an $\frac{SO(2\cdot\mathbf{5}+2)}{SU(\mathbf{5}+1)\times U(1)}$ SUSY σ-model.

In Section 2, we give a brief summary of an $SO(2N+1)$ canonical transformation, embedding of $SO(2N+1)$ group into an $SO(2N+2)$ one and fixing a $\frac{SO(2N+2)}{U(N+1)}$ coset variable. In Section 3, we recapitulate a SUSY σ-model on the coset space $\frac{SO(2N+2)}{U(N+1)}$ and its Lagrangian on the Kähler manifold, the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. The theory is invariant under a SUSY transformation and the Killing potential is expressed in terms of the coset variables. If gauge fields are introduced in the model, the theory becomes no longer invariant under the transformation. To restore the SUSY property, it is inevitable to introduce gauginos, auxiliary fields and Fayet-Ilipoulos terms, which make the theory invariant under the SUSY transformation, i.e., chiral invariant and produces a f-deformed reduced scalar potential. Optimization of the f-deformed reduced scalar potential reproduces the f-deformed solution in I. In Section 4, we show that the optimized f-deformed solution satisfies the idempotency relation $\lessdot W \gtrdot_{f\min}^2 = \lessdot W \gtrdot_{f\min}$ for a factorized density matrix $\lessdot W \gtrdot_{f\min}$. We also present a vacuum function for bosonized fermions in terms of the Nambu-Goldstone condensate and $U(1)$ phase. In Section 5, we construct an anomaly-free $\frac{SO(2N+2)}{U(N+1)}$ SUSY σ-model and see what subjects are new. We give an invariant Killing potential which is exactly derived for $SU(N+1)$ tensors. In Section 6, we give a new f-deformed reduced scalar potential. After optimization of the f-deformed reduced scalar

potential, we find an interesting f-deformed solution for an anomaly-free $\frac{SO(2\cdot\mathbf{5}+2)}{SU(\mathbf{5}+1)\times U(1)}$ SUSY σ-model which has a very different aspect from the previous solution. Finally in Section 7, we give discussions and some concluding remarks.

2 Brief summary of embedding of $SO(2N{+}1)$ Bogoliubov transformation into $SO(2N{+}2)$ group

Following I we give a brief summary of the $SO(2N{+}1)$ canonical transformation, of embedding of $SO(2N{+}1)$ group into an $SO(2N{+}2)$ group and of fixing a $\frac{SO(2N+2)}{U(N+1)}$ coset variable. Let c_α and $c^\dagger_\alpha$, $\alpha=1,\ldots,N$, be fermion annihilation and creation operators satisfying the canonical anti-commutation relations $\{c_\alpha, c^\dagger_\beta\}=\delta_{\alpha\beta}$ and $\{c^\dagger_\alpha, c^\dagger_\beta\}=\{c_\alpha, c_\beta\}=0$. The set of fermion operators $c_\alpha, c^\dagger_\alpha$, $E^\alpha{}_\beta = c^\dagger_\alpha c_\beta - 1/2 \cdot \delta_{\alpha\beta}$, $E^{\alpha\beta}=c^\dagger_\alpha c^\dagger_\beta$ and $E_{\alpha\beta}=c_\alpha c_\beta$ form an $SO(2N{+}1)$ Lie algebra. The $SO(2N{+}1)$ Lie algebra of the fermion operators contains the $U(N)(=\{E^\alpha{}_\beta\})$ and the $SO(2N)(=\{E^\alpha{}_\beta,\ E^{\alpha\beta},\ E_{\alpha\beta}\})$ Lie algebras of the pair operators as subalgebras.

An $SO(2N)$ canonical transformation $U(g)$ belongs to the fermion $SO(2N)$ Lie operators. The transformation $U(g)$ is the generalized Bogoliubov transformation [4] specified by an $SO(2N)$ matrix g

$$U(g)(c, c^\dagger)U^\dagger(g) = (c, c^\dagger)g, \quad g^\dagger g = g g^\dagger = 1_{2N}, \quad \det g = 1, \tag{2.1}$$

$$U(g)U(g') = U(gg'), \quad U(g^{-1}) = U^{-1}(g) = U^\dagger(g), \quad U(1_{2N}) = \mathbb{I}. \tag{2.2}$$

$(c,\, c^\dagger)$ is a $2N$-dimensional row vector $((c_\alpha),\, (c^\dagger_\alpha))$. $a=(a^\alpha{}_\beta)$ and $b=(b_{\alpha\beta})$ are $N\times N$ matrices. The HB $(SO(2N))$ wave function $|\, g>$ is generated as $|\, g> = U(g)|\, 0>$ ($|\, 0>$: the vacuum satisfying $c_\alpha|\, 0>=0$). The wave function $|\, g>$ is expressed as

$$|\, g> = <0\,|U(g)|\, 0> \exp(1/2 \cdot q_{\alpha\beta} c^\dagger_\alpha c^\dagger_\beta)|\, 0>, \tag{2.3}$$

$$<0\,|U(g)|\, 0> = \overline{\Phi}_{00}(g) = [\det(a)]^{1/2} = \left[\det(1_N + q^\dagger q)\right]^{-1/4} e^{i\tau/2}\ , \tag{2.4}$$

$$q = ba^{-1} = -q^{\mathrm{T}}, \text{(variable of the } SO(2N)/U(N) \text{ coset space)}, \tau = i/2 \ln[\det(a^*)/\det(a)]\,. \tag{2.5}$$

The symbols det and T denote the determinant and transposition, respectively. The overline denotes the complex conjugation.

The canonical anti-commutation relation gives us not only the above two Lie algebras but also a third algebra. Let n be the fermion number operator $n = c^\dagger_\alpha c_\alpha$. The operator $(-1)^n$ anticommutes with c_α and $c^\dagger_\alpha$; $\{c_\alpha, (-1)^n\} = \{c^\dagger_\alpha, (-1)^n\} = 0$. Let us introduce an operator Θ as $\Theta = \theta_\alpha c^\dagger_\alpha - \overline{\theta}_\alpha c_\alpha$. Here we use the summation convention over repeated indices. Due to the relation $\Theta^2 = -\overline{\theta}_\alpha \theta_\alpha$, we have

$$\left.\begin{aligned} &e^\Theta = Z + X_\alpha c^\dagger_\alpha - \overline{X}_\alpha c_\alpha, \quad \overline{X}_\alpha X_\alpha + Z^2 = 1, \\ &Z = \cos\theta, \quad X_\alpha = \theta_\alpha/\theta \sin\theta, \quad \theta^2 = \overline{\theta}_\alpha \theta_\alpha. \end{aligned}\right\} \tag{2.6}$$

From the anti-commutator of $(-1)^n$ with c_α and $c^\dagger_\alpha$ and (2.6), we obtain

$$e^\Theta (c_\alpha, c^\dagger_\alpha, \frac{1}{\sqrt{2}})(-1)^n e^{-\Theta} = (c_\beta, c^\dagger_\beta, \frac{1}{\sqrt{2}})(-1)^n \begin{bmatrix} \delta_{\beta\alpha} - \bar{X}_\beta X_\alpha & \bar{X}_\beta \bar{X}_\alpha & -\sqrt{2}Z\bar{X}_\beta \\ X_\beta X_\alpha & \delta_{\beta\alpha} - X_\beta \bar{X}_\alpha & \sqrt{2}ZX_\beta \\ \sqrt{2}ZX_\alpha & -\sqrt{2}Z\bar{X}_\alpha & 2Z^2 - 1 \end{bmatrix}. \tag{2.7}$$

From (2.1), (2.7) and the commutator of $U(g)$ with $(-1)^n$, we obtain

$$U(G)(c_\alpha, c^\dagger_\alpha, 1/\sqrt{2})(-1)^n U^\dagger(G) = (c_\beta, c^\dagger_\beta, 1/\sqrt{2})(-1)^n \begin{bmatrix} A_{\beta\alpha} & \overline{B}_{\beta\alpha} & -\overline{x}_\beta/\sqrt{2} \\ B_{\beta\alpha} & \overline{A}_{\beta\alpha} & x_\beta/\sqrt{2} \\ y_\alpha/\sqrt{2} & -\overline{y}_\alpha/\sqrt{2} & z \end{bmatrix}, \tag{2.8}$$

$$\left.\begin{aligned}&A_{\alpha\beta}=a_{\alpha\beta}-\overline{X}_\alpha Y_\beta=a_{\alpha\beta}-\overline{x}_\alpha y_\beta/2(1+z), \quad B_{\alpha\beta}=b_{\alpha\beta}+X_\alpha Y_\beta=b_{\alpha\beta}+x_\alpha y_\beta/2(1+z),\\ &x_\alpha=2ZX_\alpha,\ y_\alpha=2ZY_\alpha,\ z=2Z^2-1,\ Y_\alpha=X_\beta a^\beta_{\ \alpha}-\bar{X}_\beta b_{\beta\alpha}.\end{aligned}\right\} \quad (2.9)$$

Equation (2.8) can be written as

$$U(G)(c,c^\dagger,1/\sqrt{2})U^\dagger(G)=(c,c^\dagger,1/\sqrt{2})(z-\rho)G,\ \rho=x_\alpha c^\dagger_\alpha-\overline{x}_\alpha c_\alpha,\ \rho^2=-\overline{x}_\alpha x_\alpha=z^2-1, \quad (2.10)$$

$$G\stackrel{\mathrm{def}}{=}\begin{bmatrix} A & \overline{B} & -\overline{x}/\sqrt{2} \\ B & \overline{A} & x/\sqrt{2} \\ y/\sqrt{2} & -\overline{y}/\sqrt{2} & z \end{bmatrix},\ G^\dagger G=GG^\dagger=1_{2N+1},\ \det G=1, \quad (2.11)$$

$$U(G)U(G')=U(GG'),\ U(G^{-1})=U^{-1}(G)=U^\dagger(G),\ U(1_{2N+1})=\mathbb{I}_G. \quad (2.12)$$

$(c,\ c^\dagger,\ 1/\sqrt{2})$ is a $(2N+1)$-dimensional row vector $((c_\alpha),\ (c^\dagger_\alpha),\ 1/\sqrt{2})$. $A=(A^\alpha_{\ \beta})$ and $B=(B_{\alpha\beta})$ are $N\times N$ matrices. The transformation $U(G)$ is a nonlinear transformation with a gauge factor $z-\rho$ [10]. The $SO(2N+1)$ canonical transformation $U(G)$ is generated by the fermion $SO(2N+1)$ Lie operators. The transformation $U(G)$ is an extension of the generalized Bogoliubov transformation $U(g)$ [4] to a nonlinear transformation and is specified by the $SO(2N+1)$ matrix G.

The $SO(2N+1)$ wave function [13, 21] $|\,G>=U(G)|\,0>$ is expressed as

$$|\,G>=<0\,|U(G)|\,0>(1+r_\alpha c^\dagger_\alpha)\exp(1/2\cdot q_{\alpha\beta}c^\dagger_\alpha c^\dagger_\beta)|\,0>,\quad r_\alpha=\frac{1}{1+z}(x_\alpha+q_{\alpha\beta}\overline{x}_\beta), \quad (2.13)$$

$$<0\,|\,U(G)\,|\,0>=\overline{\Phi}_{00}(G)=\sqrt{\frac{1+z}{2}}\left[\det(1_N+q^\dagger q)\right]^{1/4}e^{i\tau/2}. \quad (2.14)$$

The $SO(2N+1)$ group is embedded into an $SO(2N+2)$ group. The embedding leads us to an unified formulation of the $SO(2N+1)$ regular representation in which paired and unpaired modes are treated in an equal way. Define $(N+1)\times(N+1)$ matrices $\mathcal{A}$ and $\mathcal{B}$ as

$$\mathcal{A}=\begin{bmatrix} A & -\overline{x}/2 \\ y/2 & (1+z)/2 \end{bmatrix},\quad \mathcal{B}=\begin{bmatrix} B & x/2 \\ -y/2 & (1-z)/2 \end{bmatrix},\quad y=x^{\mathrm{T}}a-x^\dagger b. \quad (2.15)$$

Imposing the ortho-normalization of the G, matrices $\mathcal{A}$ and $\mathcal{B}$ satisfy the ortho-normalization condition and then form an $SO(2N+2)$ matrix $\mathcal{G}$ represented as [13]

$$\mathcal{G}=\begin{bmatrix} \mathcal{A} & \overline{\mathcal{B}} \\ \mathcal{B} & \overline{\mathcal{A}} \end{bmatrix},\quad \mathcal{G}^\dagger\mathcal{G}=\mathcal{G}\mathcal{G}^\dagger=1_{2N+2},\quad \det\mathcal{G}=1. \quad (2.16)$$

By using (2.9), the matrices $\mathcal{A}$ and $\mathcal{B}$ can be decomposed as

$$\mathcal{A}=\begin{bmatrix} 1_N-\overline{x}r^{\mathrm{T}}/2 & -\overline{x}/2 \\ (1+z)r^{\mathrm{T}}/2 & (1+z)/2 \end{bmatrix}\begin{bmatrix} a & 0 \\ 0 & 1 \end{bmatrix},\ \mathcal{B}=\begin{bmatrix} 1_N+xr^{\mathrm{T}}q^{-1}/2 & x/2 \\ -(1+z)r^{\mathrm{T}}q^{-1}/2 & (1-z)/2 \end{bmatrix}\begin{bmatrix} b & 0 \\ 0 & 1 \end{bmatrix}, \quad (2.17)$$

from which we get the inverse of $\mathcal{A}$, $\mathcal{A}^{-1}$ and $SO(2N+2)/U(N+1)$ coset variable $\mathcal{Q}$ with the $N+1$-th component as

$$\mathcal{A}^{-1}=\begin{bmatrix} a^{-1} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1_N & \overline{x}/(1+z) \\ -r^{\mathrm{T}} & 1 \end{bmatrix},\quad \mathcal{Q}=\mathcal{B}\mathcal{A}^{-1}=\begin{bmatrix} q & r \\ -r^{\mathrm{T}} & 0 \end{bmatrix}=-\mathcal{Q}^{\mathrm{T}}. \quad (2.18)$$

We denote the $(N+1)$-dimension of the matrix $\mathcal{Q}$ by the index 0 and use the indices $i,\ j,\ \cdots$.

3 Recapitulation of $\frac{SO(2N+2)}{U(N+1)}$ supersymmetric σ-model and Killing potential

Following I we recapitulate a SUSY σ-model on the coset space $\frac{SO(2N+2)}{U(N+1)}$ and its Lagrangian on the Kähler manifold, the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. The simplest representation of $\mathcal{N}$=1 SUSY is a scalar multiplet $\phi=\{\mathcal{Q},\psi_L,H\}$ where $\mathcal{Q}$ and H are complex scalars and $\psi_L\equiv\frac{1+\gamma_5}{2}\psi$ is a left-handed chiral spinor. The SUSY σ-model can be constructed from the $[N]\{=N(N+1)/2\}$ scalar multiplets $\phi^{[\alpha]}=\{\mathcal{Q}^{[\alpha]},\psi_L^{[\alpha]},H^{[\alpha]}\}([\alpha]=1,\cdots,[N])$. Let the Kähler manifold be the $\frac{SO(2N+2)}{U(N+1)}$ coset manifold and redenote the complex scalar fields $\mathcal{Q}_{pq}$ as $\mathcal{Q}^{[\alpha]}([\alpha]=1,\cdots,[N])$. After eliminating the auxiliary field $H^{[\alpha]}$, the Lagrangian of a SUSY σ-model [2] is given as

$$\mathcal{L}_{\mathrm{chiral}}=-\mathcal{G}_{[\alpha][\underline{\beta}]}\left(\partial_\mu\bar{\mathcal{Q}}^{[\underline{\beta}]}\partial_\mu\mathcal{Q}^{[\alpha]}+\bar{\psi}_L^{[\underline{\beta}]}\overleftrightarrow{\boldsymbol{D}\!\!\!\!/}\,\psi_L^{[\alpha]}\right)+\frac{1}{2}\boldsymbol{R}_{[\alpha][\underline{\beta}][\gamma][\underline{\delta}]}\bar{\psi}_L^{[\underline{\beta}]}\gamma_\mu\psi_L^{[\alpha]}\bar{\psi}_L^{[\underline{\delta}]}\gamma_\mu\psi_L^{[\gamma]}. \tag{3.1}$$

The Kähler metrics admit a set of holomorphic isometries, the Killing vectors, $\mathcal{R}^{l[\alpha]}(\mathcal{Q})$ and $\overline{\mathcal{R}}^{l[\underline{\alpha}]}(\overline{\mathcal{Q}})(l=1,\cdots,\dim\mathcal{G}(\mathcal{G}\in SO(2N+2)))$, which are the solution of the Killing equation

$$\mathcal{R}^l{}_{[\underline{\beta}]}(\mathcal{Q})_{,[\alpha]}+\overline{\mathcal{R}}^l{}_{[\alpha]}(\overline{\mathcal{Q}})_{,[\underline{\beta}]}=0,\quad \mathcal{R}^l{}_{[\underline{\beta}]}(\mathcal{Q})=\mathcal{G}_{[\alpha][[\underline{\beta}]}\mathcal{R}^{l[\alpha]}(\mathcal{Q}). \tag{3.2}$$

These isometries are described geometrically by the above Killing vectors, the generators of infinitesimal coordinate transformations keeping the metric invariant: $\delta\mathcal{Q}=\mathcal{Q}'-\mathcal{Q}=\mathcal{R}(\mathcal{Q})$ and $\delta\overline{\mathcal{Q}}=\overline{\mathcal{R}}(\overline{\mathcal{Q}})$ such that $\mathcal{G}'(\mathcal{Q},\overline{\mathcal{Q}})=\mathcal{G}(\mathcal{Q},\overline{\mathcal{Q}})$. The Killing equation (3.2) is the necessary and sufficient condition for an infinitesimal coordinate transformation

$$\delta\mathcal{Q}^{[\alpha]}=\left(\delta\mathcal{B}-\delta\mathcal{A}^{\mathrm{T}}\mathcal{Q}-\mathcal{Q}\delta\mathcal{A}+\mathcal{Q}\delta\mathcal{B}^\dagger\mathcal{Q}\right)^{[\alpha]}=\widehat{\xi}_l\mathcal{R}^{l[\alpha]}(\mathcal{Q}),\quad \delta\overline{\mathcal{Q}}^{[\underline{\alpha}]}=\widehat{\xi}_l\overline{\mathcal{R}}^{l[\underline{\alpha}]}(\overline{\mathcal{Q}}), \tag{3.3}$$

where $\widehat{\xi}_l$ are infinitesimal parameters. Due to the Killing equation, the Killing vectors $\mathcal{R}^{l[\alpha]}(\mathcal{Q})$ and $\overline{\mathcal{R}}^{l[\underline{\alpha}]}(\overline{\mathcal{Q}})$ are given as the gradient of the Killing potential $\mathcal{M}^l(\mathcal{Q},\overline{\mathcal{Q}})$ such that

$$\mathcal{R}^l{}_{[\underline{\alpha}]}(\mathcal{Q})=-i\mathcal{M}^l{}_{,[\underline{\alpha}]},\quad \overline{\mathcal{R}}^l{}_{[\alpha]}(\overline{\mathcal{Q}})=i\mathcal{M}^l{}_{,[\alpha]}. \tag{3.4}$$

The Killing potential $\mathcal{M}_\sigma$ can be written as

$$\mathcal{M}_\sigma\left(\delta\mathcal{A},\delta\mathcal{B},\delta\mathcal{B}^\dagger\right)=\mathrm{Tr}\left(\delta\mathcal{G}\widetilde{\mathcal{M}_\sigma}\right)=\mathrm{tr}\left(\delta\mathcal{A}\mathcal{M}_{\sigma\delta\mathcal{A}}+\delta\mathcal{B}\mathcal{M}_{\sigma\delta\mathcal{B}^\dagger}+\delta\mathcal{B}^\dagger\mathcal{M}_{\sigma\delta\mathcal{B}}\right). \tag{3.5}$$

Trace Tr extends over the $(2N+2)\times(2N+2)$ matrices, while trace tr does over the $(N+1)\times(N+1)$ matrices. Let us introduce the $(N+1)$-dimensional matrices $\mathcal{R}(\mathcal{Q};\delta\mathcal{G})$, $\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})$ and $\mathcal{X}$ by

$$\left.\begin{array}{l}\mathcal{R}(\mathcal{Q};\delta\mathcal{G})=\delta\mathcal{B}-\delta\mathcal{A}^{\mathrm{T}}\mathcal{Q}-\mathcal{Q}\delta\mathcal{A}+\mathcal{Q}\delta\mathcal{B}^\dagger\mathcal{Q},\quad \mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})=-\delta\mathcal{A}^{\mathrm{T}}+\mathcal{Q}\delta\mathcal{B}^\dagger,\\ \mathcal{X}=(1_{N+1}+\mathcal{Q}\mathcal{Q}^\dagger)^{-1}=\mathcal{X}^\dagger.\end{array}\right\} \tag{3.6}$$

In (3.3), $\delta\mathcal{Q}=\mathcal{R}(\mathcal{Q};\delta\mathcal{G})$, Killing vector in the coset space $\frac{SO(2N+2)}{U(N+1)}$ and $\mathrm{tr}[\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})]=\mathcal{F}(\mathcal{Q})$ is derived later where $\mathcal{F}(\mathcal{Q})$ is a holomorphic function. The Killing potential $\mathcal{M}_\sigma$ is given as

$$\left.\begin{array}{l}-i\mathcal{M}_\sigma\left(\mathcal{Q},\bar{\mathcal{Q}};\delta\mathcal{G}\right)=-\mathrm{tr}\Delta\left(\mathcal{Q},\bar{\mathcal{Q}};\delta\mathcal{G}\right),\\ \Delta\left(\mathcal{Q},\bar{\mathcal{Q}};\delta\mathcal{G}\right)\stackrel{\mathrm{def}}{=}\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})-\mathcal{R}(\mathcal{Q};\delta\mathcal{G})\mathcal{Q}^\dagger\mathcal{X}=\left(\mathcal{Q}\delta\mathcal{A}\mathcal{Q}^\dagger-\delta\mathcal{A}^{\mathrm{T}}-\delta\mathcal{B}\mathcal{Q}^\dagger+\mathcal{Q}\delta\mathcal{B}^\dagger\right)\mathcal{X}.\end{array}\right\} \tag{3.7}$$

From (3.5) and (3.7), we obtain

$$-i\mathcal{M}_{\sigma\delta\mathcal{B}}=-\mathcal{X}\mathcal{Q},\quad -i\mathcal{M}_{\sigma\delta\mathcal{B}^\dagger}=\mathcal{Q}^\dagger\mathcal{X},\quad -i\mathcal{M}_{\sigma\delta\mathcal{A}}=1_{N+1}-2\mathcal{Q}^\dagger\mathcal{X}\mathcal{Q}. \tag{3.8}$$

To make clear the meaning of the Killing potential, using the $(2N+2)\times(N+1)$ isometric matrix $\mathcal{U}$ ($\mathcal{U}^{\mathrm{T}}=[\,\mathcal{B}^{\mathrm{T}},\mathcal{A}^{\mathrm{T}}\,]$, $\mathcal{U}^\dagger\mathcal{U}=1_{N+1}$), let us define the following $(2N+2)\times(2N+2)$ matrix:

$$\mathcal{W}\stackrel{\text{def}}{=}\mathcal{U}\mathcal{U}^\dagger=\left[\begin{array}{cc}\mathcal{R} & \mathcal{K}\\ -\overline{\mathcal{K}} & 1_{N+1}-\overline{\mathcal{R}}\end{array}\right]=\mathcal{W}^\dagger\ (\mathcal{W}^2=\mathcal{W}),\left\{\begin{array}{l}\mathcal{R}=\mathcal{B}\mathcal{B}^\dagger,\\ \mathcal{K}=\mathcal{B}\mathcal{A}^\dagger.\end{array}\right. \tag{3.9}$$

The $\mathcal{W}$ is a natural extension of the generalized density matrix in the $SO(2N)$ CS rep to the $SO(2N+2)$ CS rep. The matrices $\mathcal{R}$ and $\mathcal{K}$ are represented in terms of $\mathcal{Q}$ and $\mathcal{X}$ as

$$\mathcal{R}=\mathcal{Q}(1_{N+1}+\mathcal{Q}^\dagger\mathcal{Q})^{-1}\mathcal{Q}^\dagger=\mathcal{Q}\bar{\mathcal{X}}\mathcal{Q}^\dagger=1_{N+1}-\mathcal{X},\quad \mathcal{K}=\mathcal{Q}(1_{N+1}+\mathcal{Q}^\dagger\mathcal{Q})^{-1}=\mathcal{X}\mathcal{Q}. \tag{3.10}$$

Using (3.10) and (3.8), the $-i\overline{\widetilde{\widetilde{\mathcal{M}}}}_\sigma$ reads the generalized density matrix (3.9).

Introducing the gauge fields in Lagrangian (3.1), via the gauge covariant derivatives, the σ-model is no longer invariant under the SUSY transformations. To restore the SUSY, it is necessary to add the terms

$$\Delta\mathcal{L}_{\text{chiral}}=2\mathcal{G}_{[\alpha][\underline{\alpha}]}\Big(\mathcal{R}^l{}_{[\underline{\alpha}]}(\mathcal{Q})\bar{\psi}_L^{[\underline{\alpha}]}\lambda_R^l+\overline{\mathcal{R}}^l{}_{[\alpha]}(\overline{\mathcal{Q}})\overline{\lambda}_R^l\psi_L^{[\alpha]}\Big)-g_l\mathrm{tr}\big\{D^l(\mathcal{M}^l+\xi^l)\big\},\ \lambda_R^l=\frac{1-\gamma^5}{2}\lambda^l, \tag{3.11}$$

where ξ_l are Fayet-Ilipoulos parameters. Then the full Lagrangian for this model consists of the usual SUSY Yang-Mills part and the chiral part

$$\mathcal{L}=-\mathrm{tr}\left\{\frac{1}{4}\mathcal{F}^l_{\mu\nu}\mathcal{F}^l_{\mu\nu}+\frac{1}{2}\overline{\lambda}^l\not{D}\lambda^l-\frac{1}{2}D^lD^l\right\}+\mathcal{L}_{\text{chiral}}(\partial_\mu\to\boldsymbol{D}_\mu)+\Delta\mathcal{L}_{\text{chiral}}. \tag{3.12}$$

Eliminating the auxiliary field D^l by $D^l=-g_l(\mathcal{M}^l+\xi^l)$ (not summed for l), we get a RESP arising from the gauging of $SU(N+1)\times U(1)$ with a Fayet-Ilipoulos term ξ

$$\left.\begin{array}{l}V_{\text{redSC}}=\dfrac{g^2_{U(1)}}{2(N+1)}\left(\xi-i\mathcal{M}_Y\right)^2+\dfrac{g^2_{SU(N+1)}}{2}\mathrm{tr}(-i\mathcal{M}_t)^2,\\ \mathrm{tr}(-i\mathcal{M}_t)^2=\mathrm{tr}(-i\mathcal{M}_{\sigma\delta\mathcal{A}})^2-\dfrac{1}{N+1}(-i\mathcal{M}_Y)^2,\quad -i\mathcal{M}_Y=\mathrm{tr}(-i\mathcal{M}_{\sigma\delta\mathcal{A}}).\end{array}\right\} \tag{3.13}$$

To find an f-deformed solution of the SUSY σ-model, we introduce the $(N+1)$-dimensional matrices $\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})$, $\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})$ and $\mathcal{X}_f$ in the following forms:

$$\left.\begin{array}{l}\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})=\dfrac{1}{f}\delta\mathcal{B}-\delta\mathcal{A}^{\mathrm{T}}\mathcal{Q}_f-\mathcal{Q}_f\delta\mathcal{A}+f\mathcal{Q}_f\delta\mathcal{B}^\dagger\mathcal{Q}_f,\ \mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})=-\delta\mathcal{A}^{\mathrm{T}}+f\mathcal{Q}_f\delta\mathcal{B}^\dagger,\\ \mathcal{X}_f=(1_{N+1}+f^2\mathcal{Q}_f\mathcal{Q}_f^\dagger)^{-1}=\mathcal{X}^\dagger,\ \mathcal{Q}_f=\begin{bmatrix}q & \dfrac{1}{f}r_f\\ -\dfrac{1}{f}r_f^{\mathrm{T}} & 0\end{bmatrix},\ r_f=\dfrac{1}{2Z^2}\left(x+fq\overline{x}\right),\ f\stackrel{\text{def}}{=}\dfrac{1}{m_\sigma}.\end{array}\right\} \tag{3.14}$$

Due to the rescaling, the Killing potential $\mathcal{M}_\sigma$ is deformed as

$$\left.\begin{aligned}-i\mathcal{M}_{f\sigma}\big(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\big)&=-\mathrm{tr}\Delta_f\big(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\big),\\ \Delta_f\big(\mathcal{Q}_f,\bar{\mathcal{Q}}_f;\delta\mathcal{G}\big)&\stackrel{\text{def}}{=}\mathcal{R}_{fT}(\mathcal{Q}_f;\delta\mathcal{G})-\mathcal{R}_f(\mathcal{Q}_f;\delta\mathcal{G})f^2\mathcal{Q}_f^\dagger\mathcal{X}_f\\ &=\Big(f^2\mathcal{Q}_f\delta\mathcal{A}\mathcal{Q}_f^\dagger-\delta\mathcal{A}^{\mathrm{T}}-f\delta\mathcal{B}\mathcal{Q}_f^\dagger+f\mathcal{Q}_f\delta\mathcal{B}^\dagger\Big)\mathcal{X}_f,\end{aligned}\right\} \tag{3.15}$$

from which we obtain a f-deformed Killing potential $\mathcal{M}_{f\sigma}$

$$-i\mathcal{M}_{f\sigma\delta\mathcal{B}}=-f\mathcal{X}_f\mathcal{Q}_f,\quad -i\mathcal{M}_{f\sigma\delta\mathcal{B}^\dagger}=f\mathcal{Q}_f^\dagger\mathcal{X}_f,\quad -i\mathcal{M}_{f\sigma\delta\mathcal{A}}=1_{N+1}-2f^2\mathcal{Q}_f^\dagger\mathcal{X}_f\mathcal{Q}_f. \tag{3.16}$$

After the same algebraic manipulations, the inverse matrix $\mathcal{X}_f$ in (3.14) leads to the form

$$\mathcal{X}_f=\begin{bmatrix}\mathcal{Q}_{fqq^\dagger} & \mathcal{Q}_{fq\bar{r}}\\ \mathcal{Q}^\dagger_{fq\bar{r}} & \mathcal{Q}_{fr^\dagger r}\end{bmatrix},\ \chi_f=(1_N+f^2qq^\dagger)^{-1}=\chi_f^\dagger,\ \begin{array}{c}\mathcal{Q}_{fqq^\dagger}=\chi_f-Z^2\chi_f(r_fr_f^\dagger-f^2q\bar{r}_fr_f^{\mathrm{T}}q^\dagger)\chi_f,\\ \mathcal{Q}_{fq\bar{r}}=fZ^2\chi_fq\bar{r}_f,\qquad \mathcal{Q}_{fr^\dagger r}=Z^2.\end{array} \tag{3.17}$$

Substituting (3.14) and (3.17) into (3.16) and introducing a f-deformed auxiliary function $\lambda_f = r_f r_f^\dagger - f^2 q\overline{r}_f r_f^{\mathrm{T}} q^\dagger = \lambda_f^\dagger$, we can get the f-deformed Killing potential $\mathcal{M}_{f\sigma\delta\mathcal{A}}$ as

$$
\begin{aligned}
&-i\mathcal{M}_{f\sigma\delta\mathcal{A}} = \\
&\begin{bmatrix}
\begin{array}{c} 1_N - 2q^\dagger\chi_f q + 2\dfrac{Z^2}{f^2}\big(f^2 q^\dagger\chi_f\lambda_f\chi_f q + f^2 q^\dagger\chi_f q\overline{r}_f r_f^{\mathrm{T}} \\ +f^2\overline{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f q - \overline{r}_f r_f^{\mathrm{T}}\big) \end{array} & \begin{array}{c} -2\dfrac{1}{f}q^\dagger\chi_f r_f + 2\dfrac{Z^2}{f}\big(q^\dagger\chi_f\lambda_f\chi_f r_f \\ +\overline{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f r_f\big) \end{array} \\
-2\dfrac{1}{f}r_f^\dagger\chi_f q + 2\dfrac{Z^2}{f}\Big(r_f^\dagger\chi_f\lambda_f\chi_f q + r_f^\dagger\chi_f q\overline{r}_f r_f^{\mathrm{T}}\Big) & 1 - 2\dfrac{1}{f^2}r_f^\dagger\chi_f r_f + 2\dfrac{Z^2}{f^2}r_f^\dagger\chi_f\lambda_f\chi_f r_f
\end{bmatrix}.
\end{aligned}
\tag{3.18}
$$

Using $r_f = \dfrac{1}{2Z^2}(x + fq\overline{x})$ again, the following relations also can be easily proved:

$$
1 - 2\frac{1}{f^2}r_f^\dagger\chi_f r_f + 2\frac{Z^2}{f^2}r_f^\dagger\chi_f\lambda_f\chi_f r_f = \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2},
\tag{3.19}
$$

$$
\chi_f\lambda_f\chi_f r_f = \frac{1-Z^2}{Z^2}\chi_f r_f, \quad r_f^\dagger\chi_f\lambda_f\chi_f = \frac{1-Z^2}{Z^2}r_f^\dagger\chi_f, \quad q^\dagger\chi_f q = \frac{1}{f^2}(1_N - \bar{\chi}_f),
\tag{3.20}
$$

from which, we get a more compact form of the f-deformed Killing potential $\mathcal{M}_{f\sigma\delta\mathcal{A}}$ as,

$$
-i\mathcal{M}_{f\sigma\delta\mathcal{A}} = \begin{bmatrix}
1_N - 2q^\dagger\chi_f q + 2\dfrac{Z^2}{f}\Big(fq^\dagger\chi_f r_f r_f^\dagger\chi_f q - \dfrac{1}{f}\overline{\chi}_f\overline{r}_f r_f^{\mathrm{T}}\overline{\chi}_f\Big) & -2\dfrac{Z^2}{f}q^\dagger\chi_f r_f \\
-2\dfrac{Z^2}{f}r_f^\dagger\chi_f q & \dfrac{1}{f^2}(2Z^2-1) + 1 - \dfrac{1}{f^2}
\end{bmatrix}.
\tag{3.21}
$$

Owing to the rescaling, the f-deformed reduced scalar potential is written as follows:

$$
\left.
\begin{aligned}
&V_{f\mathrm{redSC}} = \frac{g_{U(1)}^2}{2(N+1)}\left(\xi - i\mathcal{M}_{fY}\right)^2 + \frac{g_{SU(N+1)}^2}{2}\mathrm{tr}(-i\mathcal{M}_{ft})^2, \\
&\mathrm{tr}(-i\mathcal{M}_{ft})^2 = \mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2 - \frac{1}{N+1}(-i\mathcal{M}_{fY})^2, \quad -i\mathcal{M}_{fY} = \mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}),
\end{aligned}
\right\}
\tag{3.22}
$$

in which each f-deformed Killing potential is computed straightforwardly as

$$
\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}) = \left(1 - 2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}\mathbf{tr}(\chi_f) + 2\frac{Z^2}{f^2}\mathbf{tr}(\chi_f r_f r_f^\dagger) - 4\frac{Z^2}{f^2}\mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f) + \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2},
\tag{3.23}
$$

$$
\begin{aligned}
\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2 &= N - 4\frac{1}{f^2}(1 - \frac{1}{f^2})N - 4\frac{1}{f^4}\mathbf{tr}(\chi_f) + 4\frac{1}{f^4}\mathbf{tr}(\chi_f\chi_f) + 4\frac{1}{f^2}(1 - \frac{1}{f^2}) <\chi_f>^* \\
&+4(1 - \frac{1}{f^2})\frac{Z^2}{f^2}\left\{\mathbf{tr}(\chi_f r_f r_f^\dagger) - \mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f)\right\} + 12\frac{Z^2}{f^4}\mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f) - 16\frac{Z^2}{f^4}\mathbf{tr}(\chi_f\chi_f r_f r_f^\dagger\chi_f) \\
&-4\frac{Z^4}{f^4}r_f^\dagger\chi_f\chi_f r_f \cdot \mathbf{tr}(\chi_f r_f r_f^\dagger) + 8\frac{Z^4}{f^4}r_f^\dagger\chi_f\chi_f r_f \cdot \mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f) \\
&+\frac{1}{f^4} + 2\frac{1}{f^2}(1 - \frac{1}{f^2})(2Z^2-1) + (1 - \frac{1}{f^2})^2 - 4\frac{Z^4}{f^4}r_f^\dagger\chi_f\chi_f r_f.
\end{aligned}
\tag{3.24}
$$

The trace $\mathbf{tr}$, taken over the $N\times N$ matrix, is used. The $r_f^\dagger\chi_f\chi_f r_f$ and $\mathbf{tr}(r_f r_f^\dagger)$ are approximately computed as

$$
\left.
\begin{aligned}
&r_f^\dagger\chi_f\chi_f r_f = \frac{1}{4Z^4}x^\dagger\chi_f x \approx \frac{1-Z^2}{Z^2} <\chi_f>, \quad <\chi_f> \stackrel{\mathrm{def}}{=} \left\{\frac{1}{N}\left[N + f^2\mathbf{tr}(q^\dagger q)\right]\right\}^{-1} = \overline{<\chi_f>}, \\
&\mathbf{tr}(r_f r_f^\dagger) = r_f^\dagger r_f = \frac{1}{4Z^4}x^\dagger\chi_f^{-1}x \approx <r_f r_f^\dagger>, \quad <r_f r_f^\dagger> \stackrel{\mathrm{def}}{=} \frac{1-Z^2}{Z^2}\frac{1}{<\chi_f>}.
\end{aligned}
\right\}
\tag{3.25}
$$

---

*We take the opportunity of pointing out a misprint in Eq. (5.12) in I [12], where the last term in the first line of (3.24) is missing.

In (3.23) and (3.24), approximating $\mathbf{tr}(\chi_f)$, $\mathbf{tr}(\chi_f r_f r_f^\dagger)$, etc. by $<\chi_f>$, $<\chi_f>\mathbf{tr}(r_f r_f^\dagger)$, etc., respectively, and using (3.25), $\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})$ and $\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2$ are computed as

$$\left.\begin{aligned}
&\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})=1+\left(1-2\frac{1}{f^2}\right)N+2\frac{1}{f^2}(2Z^2-1)<\chi_f>,\\
&\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2=1+N-4\frac{1}{f^2}\left(1-\frac{1}{f^2}\right)N\\
&\qquad -4\frac{1}{f^2}\left\{\frac{1}{f^2}(2Z^2-1)-\left(1-\frac{1}{f^2}\right)Z^2\right\}<\chi_f>+4\frac{1}{f^4}(2Z^4-1)<\chi_f>^2 .
\end{aligned}\right\} \quad (3.26)$$

Substituting (3.26) into (3.22), we obtain the f-deformed reduced scalar potential as

$$\begin{aligned}
V_{f\mathrm{redSC}}=&\frac{g_{U(1)}^2}{2(N+1)}\left[\xi+1+\left(1-2\frac{1}{f^2}\right)N+2\frac{1}{f^2}(2Z^2-1)<\chi_f>\right]^2\\
&+2\frac{g_{SU(N+1)}^2}{N+1}\frac{1}{f^2}\left[\frac{1}{f^2}N-\left\{\left(1-\frac{1}{f^2}\right)N+\left(1+3\frac{1}{f^2}\right)\right\}Z^2<\chi_f>\right.\\
&\left.+\left\{\left(1-\frac{1}{f^2}\right)N+\left(1+\frac{1}{f^2}\right)\right\}<\chi_f>+\frac{1}{f^2}\left\{2(N-1)Z^4+4Z^2-(N+2)\right\}<\chi_f>^2\right].
\end{aligned} \quad (3.27)$$

The variation of (3.27) with respect to Z^2 and $<\chi_f>$ reads

$$\begin{aligned}
&g_{U(1)}^2\left\{\xi+1+\left(1-2\frac{1}{f^2}\right)N+2\frac{1}{f^2}(2Z^2-1)<\chi_f>\right\}\\
&-2g_{SU(N+1)}^2\left[\frac{1}{4}\left\{\left(1-\frac{1}{f^2}\right)N+\left(1+3\frac{1}{f^2}\right)\right\}-\frac{1}{f^2}\left\{(N-1)Z^2+1\right\}<\chi_f>\right]=0,
\end{aligned} \quad (3.28)$$

$$\begin{aligned}
&g_{U(1)}^2\left[\xi+1+\left(1-2\frac{1}{f^2}\right)N+2\frac{1}{f^2}(2Z^2-1)<\chi_f>\right](2Z^2-1)\\
&-2g_{SU(N+1)}^2\left[\frac{1}{2}\left\{\left(1-\frac{1}{f^2}\right)N+\left(1+3\frac{1}{f^2}\right)\right\}Z^2-\frac{1}{2}\left(1-\frac{1}{f^2}\right)N-\frac{1}{2}\left(1+\frac{1}{f^2}\right)\right.\\
&\qquad\left.-\frac{1}{f^2}\left\{2(N-1)Z^4+4Z^2-(N+2)\right\}<\chi_f>\right]=0,
\end{aligned} \quad (3.29)$$

from which, we reach to solutions for Z^2 and $<\chi_f>$ obtained in I as

$$\left.\begin{aligned}
&Z^2=1+\frac{1-f^2}{4<\chi_f>},\quad <\chi_f>=\frac{1}{2}\frac{g_{U(1)}^2\{(2-f^2)N-1\}-g_{SU(N+1)}^2\{(1-f^2)N-2\}-g_{U(1)}^2f^2\xi}{g_{U(1)}^2+Ng_{SU(N+1)}^2},\\
&\mathbf{tr}(q^\dagger q)=-\frac{N}{f^2}\left(1-\frac{1}{<\chi_f>}\right),\quad (f^2\geq 1).
\end{aligned}\right\} \quad (3.30)$$

The third equation in (3.30) is a vacuum expectation value, i.e., $\mathbf{tr}(q^\dagger q)$, the invariant norm of the complex scalar Goldstone fields. Putting (3.30) into (3.27), the minimization of the reduced scalar potential with respect to the Fayet-Ilipoulos parameter ξ is realized as follows:

$$\left.\begin{aligned}
&V_{f\mathrm{redSC}}=\frac{1}{2}\frac{N}{N+1}\frac{g_{U(1)}^2g_{SU(N+1)}^2}{g_{U(1)}^2+Ng_{SU(N+1)}^2}\left\{\xi-\frac{2-f^2}{f^2}N+1+2\frac{1}{f^2}\frac{1}{N}\right\}^2+V_{f\mathrm{redSC}}^{\mathrm{min}},\\
&V_{f\mathrm{redSC}}^{\mathrm{min}}=2\frac{g_{SU(N+1)}^2}{N+1}\frac{1}{f^4}\left[\left\{1+\frac{(1-f^2)^2}{8}\right\}N+\frac{(1-f^2)^2}{8}-\frac{1}{N}\right],\ \xi_{\mathrm{min}}=\frac{2-f^2}{f^2}N-1-2\frac{1}{f^2}\frac{1}{N}.
\end{aligned}\right\} \quad (3.31)$$

Putting the ξ_{min} into (3.30), we have the final solutions just the same ones in I as

$$Z_{\mathrm{min}}^2=\frac{1}{2}+\frac{1}{2N<\chi_f>_{\mathrm{min}}},\quad <\chi_f>_{\mathrm{min}}=\frac{2-N(1-f^2)}{2N},\quad \mathbf{tr}(q^\dagger q)_{\mathrm{min}}=\frac{N}{f^2}\left(\frac{1}{<\chi_f>_{\mathrm{min}}}-1\right). \quad (3.32)$$

4 Density matrix derived from optimized solutions and vacuum function for bosonized fermions

Parallel to (3.9), let us introduce the following $2N \times 2N$ generalized density matrix:

$$W = \begin{bmatrix} R & K \\ -\overline{K} & 1_N - \overline{R} \end{bmatrix}, \quad \begin{array}{l} R = q\overline{\chi}q^\dagger, \\ K = \chi q, \end{array} \quad W^2 = W \text{ (idempotency relation).} \tag{4.1}$$

Define a factorized density-matrix $<W>_{f\min}$ and use the optimized f-deformed solution (3.32):

$$<W>_{f\min} \stackrel{\text{def}}{=} \begin{bmatrix} <\overline{\chi}_f>_{\min} <qq^\dagger>_{f\min} \cdot 1_N & <\chi_f>_{\min} <q>_{f\min} \cdot 1_N \\ <\overline{\chi}_f>_{\min} <\overline{q}>_{f\min} \cdot 1_N & 1_N - <\chi_f>_{\min} <\overline{q}q^{\mathrm{T}}>_{f\min} \cdot 1_N \end{bmatrix}$$

$$= \begin{bmatrix} <\chi_f>_{\min} \left(\dfrac{1}{<\chi_f>_{\min}} - 1 \right) \cdot 1_N & <\chi_f>_{\min} \sqrt{\dfrac{1}{<\chi_f>_{\min}} - 1} \cdot 1_N \\ <\chi_f>_{\min} \sqrt{\dfrac{1}{<\chi_f>_{\min}} - 1} \cdot 1_N & 1_N - <\chi_f>_{\min} \left(\dfrac{1}{<\chi_f>_{\min}} - 1 \right) \cdot 1_N \end{bmatrix}. \tag{4.2}$$

In (4.2) the quantities $<qq^\dagger>_{f\min}$ and $<q>_{f\min}$ are defined as $<qq^\dagger>_{f\min} = f^2 \mathbf{tr}(q^\dagger q)_{\min}/N$ (3.32) and $<q>_{f\min} = \sqrt{<qq^\dagger>_{f\min}}$, respectively. After calculating square of $<W>_{f\min}$, then we have

$$<W>^2_{f\min} = \begin{bmatrix} <\chi_f>_{\min} \left(\dfrac{1}{<\chi_f>_{\min}} - 1 \right) \cdot 1_N & <\chi_f>_{\min} \sqrt{\dfrac{1}{<\chi_f>_{\min}} - 1} \cdot 1_N \\ <\chi_f>_{\min} \sqrt{\dfrac{1}{<\chi_f>_{\min}} - 1} \cdot 1_N & 1_N - <\chi_f>_{\min} \left(\dfrac{1}{<\chi_f>_{\min}} - 1 \right) \cdot 1_N \end{bmatrix} = <W>_{f\min}, \tag{4.3}$$

which shows that the idempotency relation does hold. The vacuum function $\Phi_{00}(g)$ in $g \in SO(2N)$ (2.4) satisfies

$$\left(\boldsymbol{e}^{\boldsymbol{\alpha}}_{\ \boldsymbol{\beta}} + \frac{1}{2}\delta_{\alpha\beta} \right) \Phi_{00}(g) = \boldsymbol{e}_{\boldsymbol{\alpha\beta}} \Phi_{00}(g) = 0, \quad \Phi_{00}(1_{2N}) = 1, \tag{4.4}$$

where $\boldsymbol{e}^{\boldsymbol{\alpha}}_{\ \boldsymbol{\beta}}$, $\boldsymbol{e}_{\boldsymbol{\alpha\beta}}$ and $\boldsymbol{e}^{\boldsymbol{\alpha\beta}}$, given in I [12], are bosonized operators of the $SO(2N)$ fermion Lie operators.

Using the famous formula $\det(1_N + X) = \exp[\mathbf{tr} \ln(1_N + X)] = \exp\left[\sum_{n=1}^\infty \dfrac{(-1)^{n-1}}{n} \mathbf{tr}(X)^n \right]$, $\det(1_N + f^2 q^\dagger q)$ is calculated approximately as follows:

$$\det(1_N + f^2 q^\dagger q) = \exp\left[\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \mathbf{tr}\left\{ \left(f^2 q^\dagger q\right)^n \right\} \right] \approx \exp\left[\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} N^n \left\{ \frac{1}{N} f^2 \mathbf{tr}\left(q^\dagger q\right) \right\}^n \right]$$

$$= \left[\sum_{n=1}^\infty \frac{(-1)^{n-1}}{n} \left\{ N \left(\frac{1}{<\chi_f>} - 1 \right) \right\}^n \right] = \exp\left[\ln\left\{ 1 + N \left(\frac{1}{<\chi_f>} - 1 \right) \right\} \right] = 1 + N \left(\frac{1}{<\chi_f>} - 1 \right).$$

Then putting the optimized f-deformed solution (3.32) into the above, finally we obtain

$$\Phi_{00}(g) = \left[\det(1_N + f^2 q^\dagger q) \right]^{-\frac{1}{4}} e^{-i\frac{\tau}{2}} = \left[1 + N \left(\frac{1}{<\chi_f>} - 1 \right) \right]^{-\frac{1}{4}} e^{-i\frac{\tau}{2}}. \tag{4.5}$$

We should emphasize that a beautiful formula for vacuum function is explicitly derived.

5 Anomaly-free $\frac{SO(2N+2)}{U(N+1)}$ supersymmetric σ-model

The Lagrangian $\mathcal{L}_{\rm chiral}$ (3.1) is invariant under a $U(1)$ symmetry, i.e., under multiplication of the superfield $\phi^{[\alpha]}$ by a universal phase factor $e^{i\widehat{\theta}}$. According to Nibbelink and van Holten [22], the symmetry is expressed in terms of a holomorphic Killing vectror $\mathcal{R}^{[\alpha]}_{\widehat{\theta}}(\mathcal{Q})$ by the transformations

$$\left.\begin{aligned} \delta_{\widehat{\theta}}\mathcal{Q}^{[\alpha]} = \widehat{\theta}\mathcal{R}^{[\alpha]}_{\widehat{\theta}}(\mathcal{Q}) = i\widehat{\theta}q_{([\alpha])}\mathcal{Q}^{[\alpha]}, \\ \delta_{\widehat{\theta}}\psi^{[\alpha]}_{L} = \widehat{\theta}\mathcal{R}^{[\alpha]}_{\widehat{\theta},\ [\beta]}(\mathcal{Q})\psi^{[\beta]}_{L} = i\widehat{\theta}q_{([\alpha])}\psi^{[\alpha]}_{L}, \end{aligned}\right\} \tag{5.1}$$

in which the quantity $q_{([\alpha])}$ means the $U(1)$ charges of the superfields. There is a larger set of holomorphic Killing vectror $\mathcal{R}^{[\alpha]}_{\underline{i}}(\mathcal{Q})$ defining a Lie algebra with structure constants $f_{\underline{ij}}{}^{\underline{k}}$:

$$\mathcal{R}^{[\beta]}_{\underline{i}}(\mathcal{Q})\mathcal{R}^{[\alpha]}_{\underline{j},\ [\beta]}(\mathcal{Q}) - \mathcal{R}^{[\beta]}_{\underline{j}}(\mathcal{Q})\mathcal{R}^{[\alpha]}_{\underline{i},\ [\beta]}(\mathcal{Q}) = f_{\underline{ij}}{}^{\underline{k}}\mathcal{R}^{[\alpha]}_{\underline{k}}(\mathcal{Q}). \tag{5.2}$$

Then the Lagrangian (3.1) is invariant under the infinitesimal transformations generated by the derivation $\delta = \widehat{\theta}^{\underline{i}}\delta_{\underline{i}}$:

$$\left.\begin{aligned} \delta\mathcal{Q}^{[\alpha]} = \widehat{\theta}^{\underline{i}}\mathcal{R}^{[\alpha]}_{\underline{i}}(\mathcal{Q}), \quad \delta\overline{\mathcal{Q}}^{\underline{[\alpha]}} = \widehat{\theta}^{\underline{i}}\overline{\mathcal{R}}^{\underline{[\alpha]}}_{\underline{i}}(\overline{\mathcal{Q}}), \\ \delta\psi^{[\alpha]}_{L} = \widehat{\theta}^{\underline{i}}\mathcal{R}^{[\alpha]}_{\underline{i},\ [\beta]}(\mathcal{Q})\psi^{[\beta]}_{L}, \quad \delta\bar{\psi}^{\underline{[\alpha]}}_{L} = \widehat{\theta}^{\underline{i}}\overline{\mathcal{R}}^{\underline{[\alpha]}}_{\underline{i},\ \underline{[\beta]}}(\overline{\mathcal{Q}})\bar{\psi}^{\underline{[\beta]}}_{L}. \end{aligned}\right\} \tag{5.3}$$

Notice that the Kähler potential $\mathcal{K}(\mathcal{Q}^{\dagger}, \mathcal{Q})$ cannot be determined uniquely since the metric tensor $\mathcal{G}_{pq\underline{rs}}$ is invariant under transformations of the Kähler potential,

$$\mathcal{K}(\mathcal{Q}^{\dagger}, \mathcal{Q}) \rightarrow \mathcal{K}'(\mathcal{Q}^{\dagger}, \mathcal{Q}) = \mathcal{K}(\mathcal{Q}^{\dagger}, \mathcal{Q}) + \mathcal{F}(\mathcal{Q}) + \overline{\mathcal{F}}(\overline{\mathcal{Q}}). \tag{5.4}$$

Under the holomorphic transformations (5.3) the Kähler potential itself transforms as

$$\delta_{\underline{i}}\mathcal{K}(\mathcal{Q}^{\dagger}, \mathcal{Q}) = \mathcal{F}_{\underline{i}}(\mathcal{Q}) + \overline{\mathcal{F}}_{\underline{i}}(\overline{\mathcal{Q}}), \tag{5.5}$$

where $\mathcal{F}_{\underline{i}}(\mathcal{Q})$ and $\overline{\mathcal{F}}_{\underline{i}}(\overline{\mathcal{Q}})$ are analytic functions of $\mathcal{Q}$ and $\overline{\mathcal{Q}}$, respectively and related to a set of real scalar potentials $\mathcal{M}_{\underline{i}}(\mathcal{Q}, \overline{\mathcal{Q}})$ satisfying $\delta_{\underline{i}}\mathcal{M}_{\underline{i}}(\mathcal{Q}, \overline{\mathcal{Q}}) = f_{\underline{ij}}{}^{\underline{k}}\mathcal{M}_{\underline{k}}(\mathcal{Q}, \overline{\mathcal{Q}})$ as

$$\left.\begin{aligned} \mathcal{F}_{\underline{i}}(\mathcal{Q}) = \mathcal{K}(\mathcal{Q}^{\dagger}, \mathcal{Q})_{,[\alpha]}\mathcal{R}^{[\alpha]}_{\underline{i}}(\mathcal{Q}) + i\mathcal{M}_{\underline{i}}(\mathcal{Q}, \overline{\mathcal{Q}}), \\ \overline{\mathcal{F}}_{\underline{i}}(\overline{\mathcal{Q}}) = \mathcal{K}(\mathcal{Q}^{\dagger}, \mathcal{Q})_{,\underline{[\alpha]}}\overline{\mathcal{R}}^{\underline{[\alpha]}}_{\underline{i}}(\overline{\mathcal{Q}}) + i\mathcal{M}_{\underline{i}}(\mathcal{Q}, \overline{\mathcal{Q}}). \end{aligned}\right\} \tag{5.6}$$

The pure SUSY σ-models on cosets including Grassmannian models on $\frac{SU(N+M)}{[SU(N)\times SU(M)\times U(1)]}$ and models on manifolds $\frac{SO(2N)}{U(N)}$ are recognized to be anomalous, because they incorporate chiral fermions in non-trivial representations of the holonomy group [2]. The presence of the chiral anomalies in the internal symmetry (5.3) restricts the usefulness of these models for phenomenological applications. These anomalies must be removed to allow for a consistent gauging of the symmetries, e.g., the chiral $U(1)$ symmetry (5.1). It is realized by coupling additional chiral fermions to the σ-model preserving the holomorphic Killing vectors.

Generally speaking, if one constructs some quantum field theories based on pure coset models, one has with serious problems of anomalies in a holonomy group which particularly occur in pure SUSY coset models due to the additional chiral fermions. This is also the cases for our orthogonal cosets, coset $\frac{SO(2N)}{U(N)}$ and its extended coset $\frac{SO(2N+2)}{U(N+1)}$, though each spinor rep of $SO(2N)$ group and its extended $SO(2N+2)$ group is anomaly free. To construct a consistent SUSY coset model, we have to embed a coset coordinate in an anomaly-free spinor rep of $SO(2N+2)$ group and give a corresponding Kähler potential and then a Killing potential for the anomaly-free $\frac{SO(2N+2)}{U(N+1)}$ model based on a positive chiral spinor rep. To achieve such an object on the case of $SO(2N)$ group/algebra, van Holten et al. have proposed a method of constructing the Kähler potential and then the Killing potential [2]. This idea is very suggestive and useful for our present aim of constructing the corresponding Kähler potential and then the Killing potential for the case of $SO(2N+2)$ group/algebra.

According to Fukutome [21] in the $SO(2N+2)$ Lie algebra the total fermion space is irreducible to the $SO(2N+2)$ algebra and belongs to the irreducible spinor rep of the $SO(2N+2)$ group. It is well known that the dimension of irreducible spinor rep of the $SO(2N+2)$ group is 2^N, so that the $SO(2N+2)$ algebra can be accomodated in the fermion space. The Clifford algebra C_{2N+2} is defined on a space with 2^{N+1} dimensions, so that it cannot be constructed on the fermion space though the $SO(2N+2)$ algebra with irreducible spinor rep of 2^N dimensions can be accomodated in the fermion space. The operators $E^i{}_j$, E^{ij} and E_{ij} satisfy the commutation relations of the $SO(2N+2)$ Lie algebra where the indices $i, j, \ldots$ run over $N+1$ values $0, 1, \ldots, N$. The operator $E^i{}_i + \frac{N}{2} = n - \frac{1}{2}(-1)^n$ is the only operator commuting with all other operators in the $U(N+1)$ algebra. The $U(1)$-factor generator Y in $U(N)$ is defined as $Y = 2E^i{}_i = 2n - N - (-1)^n$ and the remaining $SU(N+1)$ generators $T^i{}_i$ are defined as the traceless part of $E^i{}_i$, $T^i{}_i = E^i{}_i + \frac{1}{2(N+1)} Y \delta^i{}_j$. Extending the van Holten et al.'s formula for Y^k-anomaly from the $SO(2N)$ case [2] to the $SO(2N+2)$ case, we also define a new $A_\pm(Y^k; N+1)$-anomaly as $A_\pm(Y^k; N+1) = \sum_{m=0}^{N+1} \binom{N+1}{m} \frac{1 \pm (-1)^m}{2} \{N - 2m - (-1)^m\}^k$. This formula makes an important role to caculate the $U(1)$-anomaly in the spinor rep of the $SO(2N+2)$. By using the $SO(2N+2)$ Lie operators E^{ij}, the expression (2.13) for the $SO(2N+1)$ wave function $| G >$ is converted to a form quite similar to the $SO(2N)$ wave function $| g >$ as

$$| G > = < 0 | U(G) | 0 > \exp\left(1/2 \cdot \mathcal{Q}_{ij} E^{ij}\right) | 0 >, \tag{5.7}$$

which leads to $U(G)| 0 > = U(\mathcal{G})| 0 >$ and we have used the nilpotency relation $(E^{\alpha 0})^2 = 0$. The construction of the Kähler potential and then the Killing potential for the $SO(2N+2)$ group/algebra is made parallel to the construction of those for the $SO(2N)$ group/algebra.

First according to [2] we define a matrix $\Xi(\mathcal{Q})$ and require a transformation rule as follows:

$$\Xi(\mathcal{Q}) \stackrel{\text{def}}{=} \begin{bmatrix} 1_{N+1} & 0 \\ \mathcal{Q} & 1_{N+1} \end{bmatrix}, \tag{5.8}$$

$$\Xi(\mathcal{Q}) \longrightarrow \Xi({}^{\mathcal{G}}\mathcal{Q}) = \mathcal{G}\Xi(\mathcal{Q})\widehat{H}^{-1}(\mathcal{Q};\mathcal{G}), \text{ with } \widehat{H}(\mathcal{Q};\mathcal{G}) = \begin{bmatrix} \left(\widehat{H}_+(\mathcal{Q};\mathcal{G}\right)^{-1} & \widehat{H}_0(\mathcal{Q};\mathcal{G}) \\ 0 & \widehat{H}_-(\mathcal{Q};\mathcal{G}) \end{bmatrix}, \tag{5.9}$$

where

$${}^{\mathcal{G}}\mathcal{Q} = (\mathcal{B} + \bar{\mathcal{A}}\mathcal{Q})(\mathcal{A} + \bar{\mathcal{B}}\mathcal{Q})^{-1} = (\mathcal{A}^{\mathrm{T}} - \mathcal{Q}\mathcal{B}^\dagger)^{-1}(\mathcal{B}^{\mathrm{T}} - \mathcal{Q}\mathcal{A}^\dagger), \text{ (due to } \mathcal{G}^{-1} = \mathcal{G}^\dagger). \tag{5.10}$$

The ${}^{\mathcal{G}}\mathcal{Q}$ is a nonlinear Möbius transformation and ${}^{\mathcal{G}'}({}^{\mathcal{G}}\mathcal{Q})={}^{\mathcal{G}'\mathcal{G}}\mathcal{Q}$ under the composition of two transformations $\mathcal{G}'$ and $\mathcal{G}$. Under the action of the $SO(2N+2)$ matrix $\mathcal{G}$ (2.16) on $\Xi(\mathcal{Q})$ from the left and that of the matrix $\widehat{H}(\mathcal{Q};\mathcal{G})^{-1}$ from the right, the $\widehat{H}(\mathcal{Q};\mathcal{G})$ takes the form

$$\widehat{H}(\mathcal{Q};\mathcal{G})=\begin{bmatrix}\left(\widehat{H}_{+}(\mathcal{Q};\mathcal{G})\right)^{-1} & \widehat{H}_0(\mathcal{Q};\mathcal{G})\\ 0 & \widehat{H}_{-}(\mathcal{Q};\mathcal{G})\end{bmatrix}=\begin{bmatrix}\mathcal{A}+\bar{\mathcal{B}}\mathcal{Q} & \bar{\mathcal{B}}\\ 0 & (\mathcal{A}^{\mathrm{T}}-\mathcal{Q}\mathcal{B}^{\dagger})^{-1}\end{bmatrix},\ \widehat{H}_{+}(\mathcal{Q};\mathcal{G})=\widehat{H}_{-}^{\mathrm{T}}(\mathcal{Q};\mathcal{G}). \quad (5.11)$$

Then we have $\det\widehat{H}_{+}(\mathcal{Q};\mathcal{G})=\det\widehat{H}_{-}(\mathcal{Q};\mathcal{G})$. Multiplying $\mathcal{G}'$ by $\mathcal{G}$, we have the relations

$$\widehat{H}_{-}(\mathcal{Q};\mathcal{G}'\mathcal{G})=\widehat{H}_{-}({}^{\mathcal{G}}\mathcal{Q};\mathcal{G}')\widehat{H}_{-}(\mathcal{Q};\mathcal{G}),\quad \widehat{H}_{+}(\mathcal{Q};\mathcal{G}'\mathcal{G})=\widehat{H}_{+}(\mathcal{Q};\mathcal{G})\widehat{H}_{+}({}^{\mathcal{G}}\mathcal{Q};\mathcal{G}'). \quad (5.12)$$

Here we redefine the Kähler potential as $\mathcal{K}(\mathcal{Q},\overline{\mathcal{Q}})=\ln\det\left(1_{N+1}+\mathcal{Q}\overline{\mathcal{Q}}\right)$. Under the nonlinear transformation (5.10), the Kähler potential transforms as

$$\mathcal{K}({}^{\mathcal{G}}\mathcal{Q},{}^{\mathcal{G}}\overline{\mathcal{Q}})=\mathcal{K}(\mathcal{Q},\overline{\mathcal{Q}})+\mathcal{F}(\mathcal{Q};\mathcal{G})+\overline{\mathcal{F}}(\overline{\mathcal{Q}};\mathcal{G}), \quad (5.13)$$

which holds for any coordinate $\mathcal{Q}$ and any frame $\mathcal{G}$. Then we have an approximate relation

$$\mathcal{F}(\mathcal{Q};\mathcal{G})=\ln\det\widehat{H}_{-}(\mathcal{Q};\mathcal{G})=-\ln\det\left[\mathcal{A}^{\mathrm{T}}-\mathcal{Q}\mathcal{B}^{\dagger}\right]=-\mathrm{tr}\ln\left[\mathcal{A}^{\mathrm{T}}-\mathcal{Q}\mathcal{B}^{\dagger}\right]\approx\mathrm{tr}[\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})], \quad (5.14)$$

where we have expanded $\mathcal{A}^{\mathrm{T}}$ and $\mathcal{B}^{\dagger}$ to the first order in the infinitesimal paramerters $\delta\mathcal{A}^{\mathrm{T}}$ and $\delta\mathcal{B}^{\dagger}$ and have used the second equation of (3.6).

Next we add matter superfields to extend the model on which an isometry group is realized with a representation chosen to cancel the anomalies. A matter representation of the isometry group is constructed by complex bundles defined on a Kähler manifold by sets of complex fields with the transformation (5.3) for chiral fermions under isometries. If one requires anomaly cancellations with the matter superfields, one may change an assignment of $U(1)$ charges by introducing a complex line bundle $\mathcal{S}$. This bundle can be defined as a complex scalar matter field coupled to the SUSY σ-model, with the infinitesimal transformation law

$$\delta_{\underline{i}}\mathcal{S}^{\lambda}=\lambda\mathcal{F}_{\underline{i}}(\mathcal{Q})\mathcal{S}. \quad (5.15)$$

For a tensor representation of the isometry group $\mathcal{T}^{\alpha_1\cdots\alpha_p}\equiv\mathcal{S}^{\lambda}T^{\alpha_1\cdots\alpha_p}$, the new field $\mathcal{T}$ obeys the transformation rule

$$\delta_{\underline{i}}\mathcal{T}^{\alpha_1\cdots\alpha_p}=\textstyle\sum_{k=1}^{p}\mathcal{R}_{\underline{i},\ \beta}^{\alpha_k}(\mathcal{Q})\mathcal{T}^{\alpha_1\cdots\beta\ldots\alpha_p}+\lambda\mathcal{F}_{\underline{i}}(\mathcal{Q})\mathcal{T}^{\alpha_1\cdots\alpha_p}. \quad (5.16)$$

A section of a minimal line bundle over $\frac{SO(2N+2)}{U(N+1)}$ is given by

$${}^{\mathcal{G}}\mathcal{S}=\left[\det\widehat{H}_{+}(\mathcal{Q};\mathcal{G})\right]^{\frac{1}{2}}\mathcal{S}=\left[\det\widehat{H}_{-}(\mathcal{Q};\mathcal{G})\right]^{\frac{1}{2}}\mathcal{S}. \quad (5.17)$$

Suppose that $\mathcal{T}_{(p;q)}^{i_1\cdots i_p}$ is an irreducible completely antisymmetric $SU(N+1)$-tensor representation with p indices and arbitrary rescaling charge q. We abbreviate it simply as $\mathcal{T}_{(p;q)}$. By taking the completely antisymmetric tensor product of a set of $SU(N+1)$ vectors $\{\mathcal{T}_1^{i_1},\cdots,\mathcal{T}_p^{i_p}\}$ we obtain an $SU(N+1)$ tensor of rank p with rescaling charge q

$$\mathcal{T}_{(p;q)}^{i_1\cdots i_p}\equiv\frac{1}{p!}\mathcal{S}^{q}T_1^{[i_1}*\cdots*T_p^{i_p]}, \quad (5.18)$$

where $[\cdots]$ denotes the completely anti-symmetrization of the indices inside the brackets. Thus we obtain a transformation of tensor $\mathcal{T}_{(p;q)}^{i_1\cdots i_p}$ as

$${}^{\mathcal{G}}\mathcal{T}_{(p;q)}^{i_1\cdots i_p}=\left[\det\widehat{H}_{-}(\mathcal{Q};\mathcal{G})\right]^{\frac{q}{2}}\left[\widehat{H}_{-}(\mathcal{Q};\mathcal{G})\right]^{i_1}_{\ \ j_1}\cdots\left[\widehat{H}_{-}(\mathcal{Q};\mathcal{G})\right]^{i_p}_{\ \ j_p}\mathcal{T}_{(p;q)}^{j_1\cdots j_p}. \quad (5.19)$$

The invariant Kähler potential for a tensor is given by

$$\mathcal{K}_{(p;q)} = \overline{\mathcal{T}}_{(p;q)j_1\cdots j_p} \mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p} \mathcal{T}^{i_1\cdots i_p}_{(p;q)}, \ \mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p} \equiv \frac{1}{p!} [\det \mathcal{X}]^{\frac{q}{2}} \mathcal{X}^{j_1}{}_{i_1} \cdots \mathcal{X}^{j_p}{}_{i_p}. \tag{5.20}$$

An $SU(N+1)$ dual tensor $\mathcal{T}_{(\overline{N+1-p};q)i_{p+1}\cdots i_{N+1}}$ with $(N+1-p)$ indices and rescaling charge q is

$$\mathcal{T}_{(\overline{N+1-p};q)i_{p+1}\cdots i_{N+1}} \equiv \frac{1}{p!} \mathcal{T}^{i_p\cdots i_1}_{(p;q)} \epsilon_{i_1\cdots i_{N+1}}, \ (\epsilon_{i_1\cdots i_{N+1}} : SU(N+1) \text{Levi-Civita tensor}) \tag{5.21}$$

which transforms under the nonlinear Möbius transformation (5.10) as

$${}^{\mathcal{G}}\mathcal{T}_{(\overline{p};q)i_1\cdots i_p} = \mathcal{T}_{(\overline{p};q)j_1\cdots j_p} \left[\widehat{H}^{-1}_{-}(\mathcal{Q};\mathcal{G})\right]^{j_1}{}_{i_1} \cdots \left[\widehat{H}^{-1}_{-}(\mathcal{Q};\mathcal{G})\right]^{j_p}{}_{i_p} \left[\det \widehat{H}_{-}(\mathcal{Q};\mathcal{G})\right]^{1+\frac{q}{2}}. \tag{5.22}$$

The invariant Kähler potential for a dual tensor is given by

$$\mathcal{K}_{(\overline{p};q)} = \mathcal{T}_{(\overline{p};q)i_1\cdots i_p} \mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p} \overline{\mathcal{T}}^{j_1\cdots j_p}_{(\overline{p};q)}, \ \mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p} \equiv \frac{1}{p!} [\det \mathcal{X}]^{1+\frac{q}{2}} [\mathcal{X}^{-1}]^{i_1}{}_{j_1} \cdots [\mathcal{X}^{-1}]^{i_p}{}_{j_p}. \tag{5.23}$$

The contributions of the invariant Kähler potentials $\mathcal{K}_{(p;q)}$ and $\mathcal{K}_{(\overline{p};q)}$ to the Killing potentials, $\mathcal{M}_{(p;q)}(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G})$ and $\mathcal{M}_{(\overline{p};q)}(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G})$ for a tensor $\mathcal{T}_{(\overline{p};q)}$ and a dual tensor $\overline{\mathcal{T}}_{(\overline{p};q)}$ of rank p with a rescaling charge q, are obtained from (5.6) to satisfy $\mathcal{F}_i(\mathcal{Q})=0$ and $\overline{\mathcal{F}}_i(\overline{\mathcal{Q}})=0$ as

$$-i\mathcal{M}_{((\frac{p}{\overline{p}});q)}(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G}) = \mathcal{K}_{((\frac{p}{\overline{p}});q),\ [\alpha]}(\mathcal{Q},\overline{\mathcal{Q}})\mathcal{R}^{[\alpha]}(\mathcal{Q}), \tag{5.24}$$

where the infinitesimal transformations generated by the derivation $\delta_{\underline{i}}\phi^{[\alpha]} = \mathcal{R}^{[\alpha]}_{\underline{i}}(\mathcal{Q})$ denote the Killing vectors $\delta\mathcal{Q}=\mathcal{R}(\mathcal{Q};\delta\mathcal{G})$ and the fields $\mathcal{T}$ obey the transformation rules

$$\delta\mathcal{T}^{i_1\cdots i_p}_{(p;q)} = \sum_{r=1}^{p} [\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})]^{i_r}{}_j\, \mathcal{T}^{i_1\cdots j\cdots i_p}_{(p;q)} + \frac{q}{2}\mathrm{tr}\,[\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})]\, \mathcal{T}^{i_1\cdots i_p}_{(p;q)}, \tag{5.25}$$

$$\delta\mathcal{T}_{(\overline{p};q)i_1\cdots i_p} = \sum_{r=1}^{p} \mathcal{T}_{(\overline{p};q)i_1\cdots j\cdots i_p} [-\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})]^{j}{}_{j_r} + \left(1+\frac{q}{2}\right) \mathrm{tr}\,[\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})]\, \mathcal{T}_{(\overline{p};q)i_1\cdots i_p}, \tag{5.26}$$

where we have used (5.14) with $\lambda=q/2$ in (5.25) and (5.16) with $\lambda=1+q/2$ in (5.26), respectively. According to van Holten et al. [2], the transformation rules (5.25) and (5.26) also can be derived by expanding the finite transformations (5.19) and (5.22) to the first order in the infinitesimal parameters $\delta\mathcal{A}^{\mathrm{T}}$ and $\delta\mathcal{B}^{\dagger}$. As an example we demonstrate the following variation:

$$\delta\left[\widehat{H}^{-1}_{-}(\mathcal{Q};\mathcal{G})\right]^{i_r}{}_{j_r} = \left[\left(\mathcal{A}^{\mathrm{T}}-\mathcal{Q}\mathcal{B}^{\dagger}\right)^{-1}\right]^{j}{}_{j_r} \left(-\delta\mathcal{A}^{\mathrm{T}}+\mathcal{Q}\delta\mathcal{B}^{\dagger}\right)^{j}{}_{i} \left[\left(\mathcal{A}^{\mathrm{T}}-\mathcal{Q}\mathcal{B}^{\dagger}\right)^{-1}\right]^{i_r}{}_{i} \approx [\mathcal{R}_T(\mathcal{Q};\delta\mathcal{G})]^{j_r}{}_{i_r}. \tag{5.27}$$

Equation (5.24) is an extended form of (5.6)($\mathcal{F}_i=\overline{\mathcal{F}}_i=0$) to tensors and its final expression reads

$$\begin{aligned} -i\mathcal{M}_{(p;q)}(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G}) &= \frac{1}{p!}\overline{\mathcal{T}}_{(p;q)j_1\cdots j_p} \\ &\times [\det \mathcal{X}]^{\frac{q}{2}} \mathcal{X}^{j_1}{}_{k_1} \cdots \mathcal{X}^{j_p}{}_{k_p} \\ &\times \left\{ \sum_{r=1}^{p} \delta^{k_1}{}_{i_1} \cdots \left[\Delta(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G})\right]^{k_r}{}_{i_r} \cdots \delta^{k_p}{}_{i_p} + \frac{q}{2}\mathrm{tr}\left[\Delta(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G})\right] \delta^{k_1}{}_{i_1} \cdots \delta^{k_p}{}_{i_p} \right\} \mathcal{T}^{i_1\cdots i_p}_{(p;q)}, \end{aligned} \tag{5.28}$$

$$\begin{aligned} -i\mathcal{M}_{(\overline{p};q)}(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G}) &= \frac{1}{p!}\mathcal{T}_{(\overline{p};q)j_1\cdots j_p} \\ &\times \left\{ \sum_{r=1}^{p} \delta^{j_1}{}_{k_1} \cdots [-\Delta(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G})]^{j_r}{}_{k_r} \cdots \delta^{j_p}{}_{k_p} + \left(1+\frac{q}{2}\right) \mathrm{tr}\left[\Delta(\mathcal{Q},\overline{\mathcal{Q}};\delta\mathcal{G})\right] \delta^{j_1}{}_{k_1} \cdots \delta^{j_p}{}_{k_p} \right\} \\ &\times [\det \mathcal{X}]^{1+\frac{q}{2}} [\mathcal{X}^{-1}]^{k_1}{}_{i_1} \cdots [\mathcal{X}^{-1}]^{k_p}{}_{i_p} \overline{\mathcal{T}}^{i_1\cdots i_p}_{(\overline{p};q)}. \end{aligned} \tag{5.29}$$

The derivation of equations (5.28) and (5.29) is made in the following way: From (5.24) and (5.20) the Killing potential for tensors is given as

$$
\begin{aligned}
-i\mathcal{M}_{(p;q)} &= \mathcal{K}_{(p;q),\ [i]}\mathcal{R}^{[i]} = \overline{\mathcal{T}}_{(p;q)j_1\cdots j_p,\ [i]}\mathcal{R}^{[i]}\mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p}\mathcal{T}^{i_1\cdots i_p}_{(p;q)} \\
&\quad + \overline{\mathcal{T}}_{(p;q)j_1\cdots j_p}\mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p,\ [i]}\mathcal{R}^{[i]}\mathcal{T}^{i_1\cdots i_p}_{(p;q)} + \overline{\mathcal{T}}_{(p;q)j_1\cdots j_p}\mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p}\mathcal{T}^{i_1\cdots i_p}_{(p;q),\ [i]}\mathcal{R}^{[i]},\ \ ([i]=(i\hat{i})),
\end{aligned}
\tag{5.30}
$$

in which $\mathcal{Q}^{[i]}$ means the $(i\hat{i})$ element of the matrix $\mathcal{Q}$ ($\hat{i}$: another component different from i), i.e., $\mathcal{Q}_{i\hat{i}}$ and $\mathcal{R}$ are given by the Killing vector, i.e., $\mathcal{R} = \delta\mathcal{Q}$ (3.3). The variation of $\delta\mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p}$ is calculated as

$$
\begin{aligned}
\delta\mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p} &= -\frac{q}{2}\frac{1}{p!}[\det\mathcal{X}]^{\frac{q}{2}}\mathrm{tr}\big\{\mathcal{X}\big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)\big\}\,\mathcal{X}^{j_1}{}_{i_1}\cdots\mathcal{X}^{j_p}{}_{i_p} \\
&\quad -\frac{1}{p!}[\det\mathcal{X}]^{\frac{q}{2}}\textstyle\sum_{r=1}^{p}\mathcal{X}^{j_1}{}_{i_1}\cdots\big\{\mathcal{X}\big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)\mathcal{X}\big\}^{j_r}{}_{i_r}\cdots\mathcal{X}^{j_p}{}_{i_p},
\end{aligned}
\tag{5.31}
$$

together with $\delta\det\mathcal{X} = -\det\mathcal{X}\cdot\mathrm{tr}\big\{\mathcal{X}\big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)\big\}$ and $\delta\mathcal{X}^{j}{}_{i} = -\big\{\mathcal{X}\big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)\mathcal{X}\big\}^{j}{}_{i}$. Taking only the $\delta\mathcal{Q}$ term in (5.31), the following type of contraction is easily carried out:

$$
\begin{aligned}
\mathcal{G}^{j_1\cdots j_p}_{(p;q)i_1\cdots i_p,\ \hat{i}i}\delta\mathcal{Q}_{i\hat{i}} &= -\frac{q}{2}\frac{1}{p!}[\det\mathcal{X}]^{\frac{q}{2}}\mathrm{tr}\,(\mathcal{R}_T-\Delta)\,\mathcal{X}^{j_1}{}_{i_1}\cdots\mathcal{X}^{j_p}{}_{i_p} \\
&\quad -\frac{1}{p!}[\det\mathcal{X}]^{\frac{q}{2}}\textstyle\sum_{r=1}^{p}\mathcal{X}^{j_1}{}_{i_1}\cdots\Big\{\mathcal{X}^{j_r}{}_{i}\,(\mathcal{R}_T-\Delta)^{i}{}_{i_r}\Big\}\cdots\mathcal{X}^{j_p}{}_{i_p},
\end{aligned}
\tag{5.32}
$$

where we have used the relation $\delta\mathcal{Q}\overline{\mathcal{Q}}\mathcal{X} = \mathcal{R}_T - \Delta$ (3.6) . $\overline{\mathcal{T}}_{(p;q)j_1\cdots j_p,[i]}\mathcal{R}^{[i]} = 0$ is evident and $\mathcal{T}^{i_1\cdots i_p}_{(p;q),[i]}\mathcal{R}^{[i]}$ is already given by (5.25). Substituting these results into (5.30) we reach (5.28).

On the other hand, from (5.24) and (5.23) the Killing potential for dual tensor is given as

$$
\begin{aligned}
-i\mathcal{M}_{(\overline{p};q)} &= \mathcal{K}_{(\overline{p};q),\ [i]}\mathcal{R}^{[i]} = \mathcal{T}_{(\overline{p};q)i_1\cdots i_p,\ [i]}\mathcal{R}^{[i]}\mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p}\overline{\mathcal{T}}^{j_1\cdots j_p}_{(\overline{p};q)} \\
&\quad + \mathcal{T}_{(\overline{p};q)i_1\cdots i_p}\mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p,\ [i]}\mathcal{R}^{[i]}\overline{\mathcal{T}}^{j_1\cdots j_p}_{(\overline{p};q)} + \mathcal{T}_{(\overline{p};q)i_1\cdots i_p}\mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p}\overline{\mathcal{T}}^{j_1\cdots j_p}_{(\overline{p};q),\ [i]}\mathcal{R}^{[i]},\ \ ([i]=(i\hat{i})),
\end{aligned}
\tag{5.33}
$$

in which the variation of $\delta\mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p}$ is computed as

$$
\begin{aligned}
\delta\mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p} &= -\Big(1+\frac{q}{2}\Big)\frac{1}{p!}[\det\mathcal{X}]^{1+\frac{q}{2}}\mathrm{tr}\,\big\{\mathcal{X}\big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)\big\}\,[\mathcal{X}^{-1}]^{i_1}{}_{j_1}\cdots[\mathcal{X}^{-1}]^{i_p}{}_{j_p} \\
&\quad +\frac{1}{p!}[\det\mathcal{X}]^{1+\frac{q}{2}}\textstyle\sum_{r=1}^{p}[\mathcal{X}^{-1}]^{j_1}{}_{i_1}\cdots\big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)^{i_r}{}_{j_r}\cdots[\mathcal{X}^{-1}]^{i_p}{}_{j_p}.
\end{aligned}
\tag{5.34}
$$

together with $\delta[\mathcal{X}^{-1}]^{j}{}_{i} = \big(\delta\mathcal{Q}\overline{\mathcal{Q}}+\mathcal{Q}\delta\overline{\mathcal{Q}}\big)^{j}{}_{i}$. Picking up also only the term $\delta\mathcal{Q}$ in (5.34), the following contraction is also easily executed in the way parallel to the one made in (5.32):

$$
\begin{aligned}
\mathcal{G}^{i_1\cdots i_p}_{(\overline{p};q)j_1\cdots j_p,\ \hat{i}i}\delta\mathcal{Q}_{i\hat{i}} &= -\Big(1+\frac{q}{2}\Big)\frac{1}{p!}[\det\mathcal{X}]^{1+\frac{q}{2}}\mathrm{tr}\,(\mathcal{R}_T-\Delta)[\mathcal{X}^{-1}]^{i_1}{}_{j_1}\cdots[\mathcal{X}^{-1}]^{i_p}{}_{j_p} \\
&\quad +\frac{1}{p!}[\det\mathcal{X}]^{1+\frac{q}{2}}\textstyle\sum_{r=1}^{p}[\mathcal{X}^{-1}]^{j_1}{}_{i_1}\cdots\Big\{(\mathcal{R}_T-\Delta)^{i_r}{}_{i}\,[\mathcal{X}^{-1}]^{i}{}_{j_r}\Big\}\cdots[\mathcal{X}^{-1}]^{i_p}{}_{j_p}.
\end{aligned}
\tag{5.35}
$$

where we have used $\delta\mathcal{Q}\overline{\mathcal{Q}}\mathcal{X} = \mathcal{R}_T - \Delta$ again. $\overline{\mathcal{T}}^{j_1\cdots j_p}_{(\overline{p};q),\ [i]}\mathcal{R}^{[i]} = 0$ is also clear and $\mathcal{T}_{(\overline{p};q)i_1\cdots i_p,\ [i]}\mathcal{R}^{[i]}$ is already contributed by (5.26). Putting these results into (5.33) finally we get (5.29).

6 Solution for anomaly-free $\frac{SO(\mathbf{10}+2)}{SU(\mathbf{5}+1)\times U(1)}$ supersymmetric σ-model

The invariant Kähler potentials for a tensor and a dual tensor are given by (5.20) and (5.23), repectively. According to the formula for $A_{\pm}(Y^k; N{+}1)$-anomaly defined in the previous section, for all $N+1$ it is sufficient to consider only the positive chirality spinor rep in which all the tensors have an even number of indices. The anti-symmetric tensor with rank 2 is identified with the coordinate $\mathcal{Q}^{ij}$ of the present coset the $U(1)$ charge of which is 4. The lowest p and q are 0 and 4 for $\mathcal{K}_{(p;q)}$ and 1 and -4 for $\mathcal{K}_{(\overline{p};q)}$. Then the Kähler potential in the present anomaly-free $\frac{SO(\mathbf{10}+2)}{SU(\mathbf{5}+1)\times U(\mathbf{1})}$ SUSY σ-model is given by

$$
\begin{aligned}
\mathcal{K}(Z,\overline{Z}) &= \frac{1}{2}\mathcal{K}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f)+\mathcal{K}_{(0;4)}+\mathcal{K}_{(\overline{1};-4)} \\
&= \frac{1}{2f^2}\ln\det\mathcal{X}_f^{-1}+(\det\mathcal{X}_f)^2|h|^2+(\det\mathcal{X}_f)^{-1}\overline{\mathcal{T}}_{(\overline{1};-4)}\mathcal{X}_f^{-1}\mathcal{T}_{(\overline{1};-4)},
\end{aligned}
\tag{6.1}
$$

where the scalar components of the various $SU(\mathbf{5}+1)$ and $U(1)$ representations are denoted by $Z^\alpha{=}(\mathcal{Q}_f^{ij}, \mathcal{T}_{(\overline{1};-4)}{=}k, h)$, $(k{=}(\mathbf{k},k_0), k^\dagger k{=}1)$. In (6.1) a factor 1/2 in the first term of the R.H.S. is included so as to get the standard normalization of the kinetic in terms of the Goldstone boson fields. Then using equations (5.20) and (5.28), and (5.23) and (5.29), the full Killing potential ($N=5$) is explicitly represented as

$$
\begin{aligned}
-i\mathcal{M}_f(\mathcal{Q}_f,\overline{\mathcal{Q}}_f;\delta\mathcal{G}) &= -i\frac{1}{2}\mathcal{M}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f;\delta\mathcal{G})-i\mathcal{M}_f(\mathcal{Q}_f,\overline{\mathcal{Q}}_f;\delta\mathcal{G})_{(0;4)}-i\mathcal{M}_f(\mathcal{Q}_f,\overline{\mathcal{Q}}_f;\delta\mathcal{G})_{(\overline{1};-4)} \\
&= -\mathrm{tr}\left[\Delta_f(\mathcal{Q}_f,\overline{\mathcal{Q}}_f;\delta\mathcal{G})\right]\left(\frac{1}{2f^2}-2\mathcal{K}_{(0;4)}+\mathcal{K}_{(\overline{1};-4)}\right)-e^{f^2\mathcal{K}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f)}k^\dagger\Delta_f(\mathcal{Q}_f,\overline{\mathcal{Q}}_f;\delta\mathcal{G})\mathcal{X}_f^{-1}k \\
&= -\mathrm{tr}\left[\left(f^2\mathcal{Q}_f\delta\mathcal{A}\mathcal{Q}_f^\dagger-\delta\mathcal{A}^{\mathrm{T}}-f\delta\mathcal{B}\mathcal{Q}_f^\dagger+f\mathcal{Q}_f\delta\mathcal{B}^\dagger\right)\mathcal{X}_f\right]\left(\frac{1}{2f^2}-2\mathcal{K}_{(0;4)}+\mathcal{K}_{(\overline{1};-4)}\right) \\
&\quad -e^{f^2\mathcal{K}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f)}k^\dagger\left(f^2\mathcal{Q}_f\delta\mathcal{A}\mathcal{Q}_f^\dagger-\delta\mathcal{A}^{\mathrm{T}}-f\delta\mathcal{B}\mathcal{Q}_f^\dagger+f\mathcal{Q}_f\delta\mathcal{B}^\dagger\right)\mathcal{X}_f\mathcal{X}_f^{-1}k,
\end{aligned}
\tag{6.2}
$$

where we have used (3.14), (3.15) and $(\det\mathcal{X}_f)^{-1}=e^{f^2\mathcal{K}_{f\sigma}}$. Comparing (6.2) with the expression for the Killing potential (3.5) we obtain a f-deformed Killing potential $\mathcal{M}_{f\sigma}$ ($N=5$)

$$
\left.
\begin{aligned}
-i\mathcal{M}_{f\sigma\delta\mathcal{B}} &= -f\mathcal{X}_f\mathcal{Q}_f\left(\frac{1}{2f^2}-2\mathcal{K}_{(0;4)}+\mathcal{K}_{(\overline{1};-4)}\right)-fe^{f^2\mathcal{K}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f)}kk^\dagger\mathcal{Q}_f, \\
-i\mathcal{M}_{f\sigma\delta\mathcal{B}^\dagger} &= f\mathcal{Q}_f^\dagger\mathcal{X}_f\left(\frac{1}{2f^2}-2\mathcal{K}_{(0;4)}+\mathcal{K}_{(\overline{1};-4)}\right)+fe^{f^2\mathcal{K}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f)}\mathcal{Q}_f^\dagger kk^\dagger, \\
-i\mathcal{M}_{f\sigma\delta\mathcal{A}} &= \left(1_{N+1}-2f^2\mathcal{Q}_f^\dagger\mathcal{X}_f\mathcal{Q}_f\right)\left(\frac{1}{2f^2}-2\mathcal{K}_{(0;4)}+\mathcal{K}_{(\overline{1};-4)}\right) \\
&\qquad +e^{f^2\mathcal{K}_{f\sigma}(\mathcal{Q}_f,\overline{\mathcal{Q}}_f)}\left(\overline{k}k^{\mathrm{T}}-f^2\mathcal{Q}_f^\dagger kk^\dagger\mathcal{Q}_f\right).
\end{aligned}
\right\}
\tag{6.3}
$$

Substituting (3.14) and (3.17) into the last equation of (6.3) and using again the auxiliary function $\lambda_f=r_fr_f^\dagger-f^2q\overline{r}_fr_f^{\mathrm{T}}q^\dagger$, we get the f-deformed Killing potential $\mathcal{M}_{f\sigma\delta\mathcal{A}}$ as

$$
-i\mathcal{M}_{f\sigma\delta\mathcal{A}} =
$$

$$
\begin{bmatrix}
\begin{array}{c}
\Big\{1_N - 2q^\dagger\chi_f q + 2\dfrac{Z^2}{f^2}\big(f^2 q^\dagger\chi_f\lambda_f\chi_f q + f^2 q^\dagger\chi_f q\overline{r}_f r_f^{\mathrm{T}} \\
+ f^2\overline{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f q - \overline{r}_f r_f^{\mathrm{T}}\big)\Big\} \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big\{\overline{\mathbf{k}}\mathbf{k}^{\mathrm{T}} - (fq^\dagger\mathbf{k} - \overline{r}_f k_0)(f\mathbf{k}^\dagger q - \overline{k}_0 r_f^{\mathrm{T}})\Big\}
\end{array}
&
\begin{array}{c}
-2\dfrac{1}{f}q^\dagger\chi_f r_f + 2\dfrac{Z^2}{f}\big(q^\dagger\chi_f\lambda_f\chi_f r_f \\
+ \overline{r}_f r_f^{\mathrm{T}} q^\dagger\chi_f r_f\big) \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big\{\overline{\mathbf{k}}k_0 - (fq^\dagger\mathbf{k} - \overline{r}_f k_0)\mathbf{k}^\dagger r_f\Big\}
\end{array}
\\
\begin{array}{c}
-2\dfrac{1}{f}r_f^\dagger\chi_f q + 2\dfrac{Z^2}{f}\left(r_f^\dagger\chi_f\lambda_f\chi_f q + r_f^\dagger\chi_f q\overline{r}_f r_f^{\mathrm{T}}\right) \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big\{\overline{k}_0\mathbf{k}^{\mathrm{T}} - r_f^\dagger\mathbf{k}(f\mathbf{k}^\dagger q - \overline{k}_0 r_f^{\mathrm{T}})\Big\}
\end{array}
&
\begin{array}{c}
1 - 2\dfrac{1}{f^2}r_f^\dagger\chi_f r_f + 2\dfrac{Z^2}{f^2}r_f^\dagger\chi_f\lambda_f\chi_f r_f \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big(\overline{k}_0 k_0 - r_f^\dagger\mathbf{k}\mathbf{k}^\dagger r_f\Big)
\end{array}
\end{bmatrix}. \tag{6.4}
$$

Using the relations (3.19) and (3.20), we get a more compact form of $-i\mathcal{M}_{f\sigma\delta\mathcal{A}}$ as

$$
-i\mathcal{M}_{f\sigma\delta\mathcal{A}} =
$$

$$
\begin{bmatrix}
\begin{array}{c}
\Big\{1_N - 2q^\dagger\chi_f q + 2\dfrac{Z^2}{f}\Big(fq^\dagger\chi_f r_f r_f^\dagger\chi_f q - \dfrac{1}{f}\overline{\chi}_f\overline{r}_f r_f^{\mathrm{T}}\overline{\chi}_f\Big)\Big\} \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big\{\overline{\mathbf{k}}\mathbf{k}^{\mathrm{T}} - (fq^\dagger\mathbf{k} - \overline{r}_f k_0)(f\mathbf{k}^\dagger q - \overline{k}_0 r_f^{\mathrm{T}})\Big\}
\end{array}
&
\begin{array}{c}
-2\dfrac{Z^2}{f}q^\dagger\chi_f r_f \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big\{\overline{\mathbf{k}}k_0 - (fq^\dagger\mathbf{k} - \overline{r}_f k_0)\mathbf{k}^\dagger r_f\Big\}
\end{array}
\\
\begin{array}{c}
-2\dfrac{Z^2}{f}r_f^\dagger\chi_f q \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big\{\overline{k}_0\mathbf{k}^{\mathrm{T}} - r_f^\dagger\mathbf{k}(f\mathbf{k}^\dagger q - \overline{k}_0 r_f^{\mathrm{T}})\Big\}
\end{array}
&
\begin{array}{c}
\left\{\dfrac{1}{f^2}(2Z^2 - 1) + 1 - \dfrac{1}{f^2}\right\} \\
\times\left(\dfrac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
+ e^{f^2\mathcal{K}_{f\sigma}}\Big(\overline{k}_0 k_0 - r_f^\dagger\mathbf{k}\mathbf{k}^\dagger r_f\Big)
\end{array}
\end{bmatrix}. \tag{6.5}
$$

The Kähler potentials are given as $\mathcal{K}_{(0;4)} = (\det\mathcal{X}_f)^2|h|^2$ and $\mathcal{K}_{(\overline{1};-4)} = (\det\mathcal{X}_f)^{-1}k^\dagger\mathcal{X}_f^{-1}k$. The f-deformed reduced scalar potential is given by (3.22) in which each f-deformed Killing potential is computed straightforwardly as

$$
\begin{aligned}
\mathrm{tr}\,(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}) = &\left\{\left(1 - 2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}\mathbf{tr}(\chi_f) + 2\frac{Z^2}{f^2}\mathbf{tr}(\chi_f r_f r_f^\dagger) - 4\frac{Z^2}{f^2}\mathbf{tr}(\chi_f r_f r_f^\dagger\chi_f)\right. \\
&\left.+ \frac{1}{f^2}(2Z^2 - 1) + 1 - \frac{1}{f^2}\right\}\left(\frac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
&+ e^{f^2\mathcal{K}_{f\sigma}}\mathbf{tr}\Big\{\overline{\mathbf{k}}\mathbf{k}^{\mathrm{T}} - (fq^\dagger\mathbf{k} - \overline{r}_f k_0)(f\mathbf{k}^\dagger q - \overline{k}_0 r_f^{\mathrm{T}})\Big\} + e^{f^2\mathcal{K}_{f\sigma}}\Big(\overline{k}_0 k_0 - r_f^\dagger\mathbf{k}\mathbf{k}^\dagger r_f\Big)\,.
\end{aligned} \tag{6.6}
$$

Using approximations for $\mathbf{tr}(\chi_f)$, $\mathbf{tr}(\chi_f r_f r_f^\dagger)$ etc. and (3.25), $\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})$ is calculated as

$$
\begin{aligned}
\mathrm{tr}\,(-i\mathcal{M}_{f\sigma\delta\mathcal{A}}) = &\left\{1 + \left(1 - 2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}(2Z^2 - 1) < \chi_f > \right\}\left(\frac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\overline{1};-4)}\right) \\
&+ e^{f^2\mathcal{K}_{f\sigma}}\mathbf{tr}\Big\{\overline{\mathbf{k}}\mathbf{k}^{\mathrm{T}} - (fq^\dagger\mathbf{k} - \overline{r}_f k_0)(f\mathbf{k}^\dagger q - \overline{k}_0 r_f^{\mathrm{T}})\Big\} + e^{f^2\mathcal{K}_{f\sigma}}\Big(\overline{k}_0 k_0 - r_f^\dagger\mathbf{k}\mathbf{k}^\dagger r_f\Big),
\end{aligned} \tag{6.7}
$$

$\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2$ is also computed, though we omit the result since its expression is very lengthy.

Substituting (6.7) and the $\mathrm{tr}(-i\mathcal{M}_{f\sigma\delta\mathcal{A}})^2$ into (3.22), we get the f-deformed reduced scalar potential $V_{f\mathrm{redSC}}$ as

$$
\begin{aligned}
V_{f\mathrm{redSC}} = &\frac{g_{U(1)}^2}{2(N+1)} \cdot \Bigg[\xi + \left\{ 1 + \left(1-2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}(2Z^2-1)<\chi_f> \right\} \left(\frac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\bar{1};-4)} \right) \\
&+ e^{f^2\mathcal{K}_{f\sigma}} \left\{ 1 + N|\mathbf{k}|^2 - \left(\frac{1-Z^2}{Z^2} + N|\mathbf{k}|^2 \right) \frac{1}{<\chi_f>} \right\} \Bigg]^2 + 2\frac{g_{SU(N+1)}^2}{N+1} \cdot \Bigg[\frac{N+1}{4} \Bigg[1 + N - 4\frac{1}{f^2}\left(1-\frac{1}{f^2}\right)N \\
&- 4\frac{1}{f^2}\left\{ \frac{1}{f^2}(2Z^2-1) - \left(1-\frac{1}{f^2}\right)Z^2 \right\} <\chi_f> + 4\frac{1}{f^4}(2Z^4-1)<\chi_f>^2 \Bigg] \left(\frac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\bar{1};-4)} \right)^2 \\
&+ \Bigg[2e^{f^2\mathcal{K}_{f\sigma}} \left\{ \left(1-2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}(1-Z^2) + 2\frac{1}{f^2}(2Z^2-1)<\chi_f> \right\} \\
&\quad \times \left\{ (N+1)|\mathbf{k}|^2 - \left(\frac{1-Z^2}{Z^2}(1-|\mathbf{k}|^2) + N|\mathbf{k}|^2 \right) \frac{1}{<\chi_f>} \right\} - 8\frac{1}{f^2}(1-Z^2)|\mathbf{k}|^2 e^{f^2\mathcal{K}_{f\sigma}} \left(1 - N\frac{1}{<\chi_f>} \right) \\
&+ 2e^{f^2\mathcal{K}_{f\sigma}} \left\{ \frac{1}{f^2}(2Z^2-1) + 1 - \frac{1}{f^2} \right\} \left(1 - |\mathbf{k}|^2 - |\mathbf{k}|^2 \frac{1-Z^2}{Z^2} \frac{1}{<\chi_f>} \right) \Bigg] \left(\frac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\bar{1};-4)} \right) \\
&+ e^{2f^2\mathcal{K}_{f\sigma}} \left[1 + N(N+2)|\mathbf{k}|^4 - 2N|\mathbf{k}|^2 \left\{ \frac{1-Z^2}{Z^2} + (N+1)|\mathbf{k}|^2 \right\} \frac{1}{<\chi_f>} + N|\mathbf{k}|^2 \left\{ \frac{1-Z^4}{Z^4} + N|\mathbf{k}|^2 \right\} \frac{1}{<\chi_f>^2} \right] \\
&- \frac{1}{4} \Bigg[\left\{ 1 + \left(1-2\frac{1}{f^2}\right)N + 2\frac{1}{f^2}(2Z^2-1)<\chi_f> \right\} \left(\frac{1}{2f^2} - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\bar{1};-4)} \right) \\
&\qquad + e^{f^2\mathcal{K}_{f\sigma}} \left\{ 1 + N|\mathbf{k}|^2 - \left(\frac{1-Z^2}{Z^2} + N|\mathbf{k}|^2 \right) \frac{1}{<\chi_f>} \right\} \Bigg]^2 \Bigg] \quad (N=5).
\end{aligned} \tag{6.8}
$$

In the above we have approximated the terms $\mathbf{tr}\left(\overline{r}_f k^\dagger q\right)$, $r_f^{\mathrm{T}} q^\dagger k$, $\mathbf{tr}\left(q^\dagger k r_f^{\mathrm{T}}\right)$ and $k^\dagger q \overline{r}_f$ to be zero. We have also used the relation $|\mathbf{k}|^2 + |k_0|^2 = 1$. Then we have only magnitude of the vector $\mathbf{k}$, $|\mathbf{k}|^2$ and magnitude of k_0, $|k_0|^2$. This means that the f-deformed reduced scalar potential is manifestly invariant under an $SU(\mathbf{5})$ and a $U(1)$ transformations, respectively. Variations of (6.8) with respect to Z^2 and $<\chi_f>$ lead to the following cubic equation for $\left(1/2f^2 - 2\mathcal{K}_{(0;4)} + \mathcal{K}_{(\bar{1};-4)}\right)$ $(\equiv E)$:

$$
\begin{aligned}
&8(N+1)\frac{1}{f^6}<\chi_f>\left\{ (1-Z^2)<\chi_f> - \frac{1}{4}f^2\left(1-\frac{1}{f^2}\right) \right\} E^3 \\
&+ \frac{1}{f^4}\frac{1}{Z^4}\frac{1}{<\chi_f>} e^{f^2\mathcal{K}_{f\sigma}} \Big[-(N+1) - 8 + 4N(2-f^2) + 4\{4-f^2-N(2-f^2)\}|\mathbf{k}|^2 \\
&+ [43 - f^2 - N(13-7f^2) + \{-48 + 5N + 13N^2 + (8-7N-7N^2)f^2\}|\mathbf{k}|^2]Z^4 - 32\{1+(N-2)|\mathbf{k}|^2\}Z^6 \\
&+ 2[N+1+4(1-|\mathbf{k}|^2) - 8(1-|\mathbf{k}|^2)Z^2 - 4\{3-(3N+2)|\mathbf{k}|^2\}Z^4 + 2\{11-N+(N^2-11N-8)|\mathbf{k}|^2\}Z^6]<\chi_f> \Big] E^2 \\
&+ 2\frac{1}{f^2}\frac{1}{Z^6}\frac{1}{<\chi_f>^3} e^{2f^2\mathcal{K}_{f\sigma}} \Big[\Big[2 - 2\{2-N(2-f^2)\}|\mathbf{k}|^2 - N\{4-f^2-N(2-f^2)\}|\mathbf{k}|^4 \Big] Z^2 \\
&- 2\{2+3(N-2)|\mathbf{k}|^2\}Z^4 + 2(1-N|\mathbf{k}|^2)\{1+(N-2)|\mathbf{k}|^2\}Z^6 \\
&- 2\Big[1 - |\mathbf{k}|^2 - N|\mathbf{k}|^2 - (1-|\mathbf{k}|^2)(1-N|\mathbf{k}|^2)Z^2 - \{3-(3N+2)|\mathbf{k}|^2\}Z^4 + (1-N|\mathbf{k}|^2)\{3-(N-2)|\mathbf{k}|^2\}Z^6 \Big] <\chi_f> \\
&+ 2|\mathbf{k}|^2 Z^2 \{2N+1-(N+1)(1-2N|\mathbf{k}|^2)Z^4\} <\chi_f>^2 \Big] E \\
&+ 2N|\mathbf{k}|^2 \frac{1}{Z^4}\frac{1}{<\chi_f>^3} e^{3f^2\mathcal{K}_{f\sigma}} \left\{ (1-N|\mathbf{k}|^2)\frac{1-Z^2}{Z^2}\frac{1}{<\chi_f>} - |\mathbf{k}|^2 \right\} = 0 \quad (N=5).
\end{aligned} \tag{6.9}
$$

If only the first term in (6.9) is taken up, the solutions for Z^2 and $<\chi_f>$ are realized, which has already been obtained in I [12]. Instead of such solutions, here we seek for another solutions. For this aim, the last term is assumed to vanish: $(1-N|\mathbf{k}|^2)(1-Z^2)Z^{-2}\cdot<\chi_f>^{-1}-|\mathbf{k}|^2=0$ $(N=5)$. Due to this relation, (6.9) becomes a quadratic equation for E. It is necessary to analyze a value and especially a sign of E which brings positive definiteness of a matter-extended Kähler metric $(E>0)$ or negative kinetic-energy ghosts $(E<0)$ [2]. The Z^2 and $<\chi_f>$ are connected with each other through $|\mathbf{k}|^2$. Substituting $<\chi_f>=(1-N|\mathbf{k}|^2)|\mathbf{k}|^{-2}\cdot(1-Z^2)Z^{-2}$ derived from the above relation into the variational equation for Z^2, we obtain an optimized equation for Z^2, which we omit here since it is very lengthy. Taking only the zeroth and first order of Z^2 $(0<Z^2<1)$ in the optimized equation, we finally reach our ultimate goal of solution for Z^2 $(N=5)$. Putting this solution into $<\chi_f>=(1-N|\mathbf{k}|^2)|\mathbf{k}|^{-2}\cdot(1-Z^2)Z^{-2}$ $(N=5)$, then we have

$$
\begin{aligned}
Z^2=&\left[\!\!\left[g^2_{U(1)}\cdot\left\{4\frac{(1-N|\mathbf{k}|^2)^2}{|\mathbf{k}|^4}E+f^2e^{f^2\mathcal{K}_{f\sigma}}\right\}+2g^2_{SU(N+1)}\cdot 2\left[4e^{f^2\mathcal{K}_{f\sigma}}(1-|\mathbf{k}|^2)-\frac{(1-N|\mathbf{k}|^2)^2}{|\mathbf{k}|^4}E-\frac{1}{4}f^2e^{f^2\mathcal{K}_{f\sigma}}\right]\right]\!\!\right]\\
&\times\left[\!\!\left[g^2_{U(1)}\cdot\left[2\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\left\{f^2\xi+\left((N+1)f^2-2N+12\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\right)E\right\}-2f^2e^{f^2\mathcal{K}_{f\sigma}}\left(1+N^2|\mathbf{k}|^2-\frac{1}{|\mathbf{k}|^2}\right)\right.\right.\right.\\
&\qquad\left.+\frac{1}{2}f^2e^{f^2\mathcal{K}_{f\sigma}}\left\{f^2\xi\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\frac{1}{E}+8+((N+1)f^2-2N)\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right\}\right]\\
&+2g^2_{SU(N+1)}\cdot\frac{1}{4}\left[\left[\left\{-2(N+1)(3-f^2)\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}+8(N+1)\frac{(1-N|\mathbf{k}|^2)^2}{|\mathbf{k}|^4}\right\}E\right.\right.\\
&+16e^{f^2\mathcal{K}_{f\sigma}}\left[(N+1)f^2(1-N|\mathbf{k}|^2)+(1-|\mathbf{k}|^2)\left\{8-f^2+(2-(2-f^2)N)\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right\}+(2-f^2)\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right]\\
&+4f^4N|\mathbf{k}|^2\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\left(1-\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right)e^{2f^2\mathcal{K}_{f\sigma}}\frac{1}{E}\\
&-4\left[\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\left\{(N+1)f^2-2N+8\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\right\}E+e^{f^2\mathcal{K}_{f\sigma}}\left(1+N^2|\mathbf{k}|^2-\frac{1}{|\mathbf{k}|^2}\right)\right]\\
&\left.\left.\left.-f^2e^{f^2\mathcal{K}_{f\sigma}}\left[\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\left\{(N+1)f^2-2N+8\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\right\}-f^2e^{f^2\mathcal{K}_{f\sigma}}\frac{1-|\mathbf{k}|^2(1+N^2|\mathbf{k}|^2)}{1-N|\mathbf{k}|^2}\frac{1}{E}\right]\right]\right]\!\!\right]^{-1},
\end{aligned}
\tag{6.10}
$$

and

$$
\begin{aligned}
<\chi_f>=&\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\left[\!\!\left[g^2_{U(1)}\cdot\left[2\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\left\{f^2\xi+\left((N+1)f^2-2N+10\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\right)E\right\}\right.\right.\right.\\
&-2f^2e^{f^2\mathcal{K}_{f\sigma}}\left(1+N^2|\mathbf{k}|^2-\frac{1}{|\mathbf{k}|^2}\right)+\frac{1}{2}f^2e^{f^2\mathcal{K}_{f\sigma}}\left\{f^2\xi\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\frac{1}{E}+6+((N+1)f^2-2N)\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right\}\Bigg]\\
&+2g^2_{SU(N+1)}\cdot\frac{1}{4}\left[\left\{-2(N+1)\left(3-f^2\right)\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}+8N\frac{\left(1-N|\mathbf{k}|^2\right)^2}{|\mathbf{k}|^4}\right\}E\right.\\
&+16e^{f^2\mathcal{K}_{f\sigma}}\left[(N+1)f^2\left(1-N|\mathbf{k}|^2\right)+\left(1-|\mathbf{k}|^2\right)\left\{6-f^2+\left(2-\left(2-f^2\right)N\right)\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right\}+\left(2-f^2\right)\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right]\\
&-2f^2e^{f^2\mathcal{K}_{f\sigma}}+4f^4N|\mathbf{k}|^2\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\left(1-\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\right)e^{2f^2\mathcal{K}_{f\sigma}}\frac{1}{E}\\
&-4\left[\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\left\{(N+1)f^2-2N+8\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\right\}E+e^{f^2\mathcal{K}_{f\sigma}}\left(1+N^2|k|^2-\frac{1}{|\mathbf{k}|^2}\right)\right]\\
&\left.\left.\left.-f^2e^{f^2\mathcal{K}_{f\sigma}}\left[\frac{|\mathbf{k}|^2}{1-N|\mathbf{k}|^2}\left\{(N+1)f^2-2N+8\frac{1-N|\mathbf{k}|^2}{|\mathbf{k}|^2}\right\}-f^2e^{f^2\mathcal{K}_{f\sigma}}\frac{1-|\mathbf{k}|^2\left(1+N^2|\mathbf{k}|^2\right)}{1-N|\mathbf{k}|^2}\frac{1}{E}\right]\right]\right]\!\!\right]\\
&\times\left[\!\!\left[g^2_{U(1)}\cdot\left\{4\frac{\left(1-N|\mathbf{k}|^2\right)^2}{|\mathbf{k}|^4}E+f^2e^{f^2\mathcal{K}_{f\sigma}}\right\}+2g^2_{SU(N+1)}\cdot2\left[4e^{f^2\mathcal{K}_{f\sigma}}\left(1-|\mathbf{k}|^2\right)-\frac{\left(1-N|\mathbf{k}|^2\right)^2}{|\mathbf{k}|^4}E-\frac{1}{4}f^2e^{f^2\mathcal{K}_{f\sigma}}\right]\right]\!\!\right]^{-1}.
\end{aligned}
\tag{6.11}
$$

7 Discussions and concluding remarks

By embedding the $SO(2N+1)$ group into an $SO(2N+2)$ group and using the $\frac{SO(2N+2)}{U(N+1)}$ coset variables [13], we have investigated a new aspect of the SUSY σ-model on the Kähler manifold of the symmetric space $\frac{SO(2N+2)}{U(N+1)}$. A consistent theory of coupling of gauge- and matter-superfields to the SUSY σ-model has been proposed on the Kähler coset space. In the theory a mathematical tool for constructing the Killing potential has been given. Further we have applied the theory to the explicit construction of the SUSY σ-model on the coset space $\frac{SO(2N+2)}{U(N+1)}$. We should emphasized again that if one wants to develop some rigorous quantum-field theories based on pure coset models, inevitably one must face the very difficult problem of anomalies in a holonomy group. Such a problem particularly occurs in SUSY coset models due to the existence of the chiral fermions [2]. This is also the case for our $\frac{SO(2N+2)}{U(N+1)}$ coset model though the spinor rep of $SO(2N+2)$ group is anomaly free. But we were able to construct successfully the invariant Killing potential in the present anomaly-free SUSY σ-model which is equivalent to the so-called generalized density matrix in the Hartree-Bogoliubov theory. Its diagonal-block part is related to the present reduced scalar potential with a Fayet-Ilipoulos term.

In order to see the behaviour of the vacuum expectation value of σ-model fields, after rescaling the Goldstone fields, we have optimized the f-deformed reduced scalar potential and found interesting f-deformed solution for an anomaly-free $\frac{SO(2\cdot\mathbf{5}+2)}{SU(\mathbf{5}+1)\times U(1)}$ SUSY σ-model. The way of finding these solutions, i.e., solutions for Z^2 (6.10) and $<\chi_f>$ (6.11), is essentially different from the way of finding the previous solutions in (3.30) which are quite the same as the ones obtained in I. To observe clearly the difference in the solution forms, we consider the special condition, $|\mathbf{k}|^2=1$ ($|k_0|^2=0$), and simplify a form of the solution for Z^2 as

$$
\begin{aligned}
Z^2 = & \left[\!\!\left[g_{U(1)}^2 \cdot \left\{ 4\,(1-N)^2 E + f^2 e^{f^2 \mathcal{K}_{f\sigma}} \right\} + 2 g_{SU(N+1)}^2 \cdot 2 \left[-\,(1-N)^2 E - \frac{1}{4} f^2 e^{f^2 \mathcal{K}_{f\sigma}} \right] \right]\!\!\right] \\
& \times \left[\!\!\left[g_{U(1)}^2 \cdot \left[2\,(1-N) \{ f^2 \xi + ((N+1) f^2 + 12 - 14N) E \} \right.\right.\right. \\
& \qquad \left. -2 f^2 e^{f^2 \mathcal{K}_{f\sigma}} N^2 + \frac{1}{2} f^2 e^{f^2 \mathcal{K}_{f\sigma}} \left\{ 8 + \left(f^2 \xi \frac{1}{E} + (N+1) f^2 - 2N \right) \frac{1}{1-N} \right\} \right] \\
& + 2 g_{SU(N+1)}^2 \cdot \frac{1}{4} \left[(N+1) \left\{ -2\,(3-f^2)\,(1-N) + 8 (1-N)^2 \right\} E \right. \\
& \qquad + 16 e^{f^2 \mathcal{K}_{f\sigma}} \left[f^2 (1-N^2) + (2-f^2) \frac{1}{1-N} \right] - 4 f^4 \frac{N^2}{(1-N)^2} e^{2 f^2 \mathcal{K}_{f\sigma}} \frac{1}{E} \\
& \qquad - 4 \left[(1-N) \{ (N+1) f^2 + 8 - 10N \} E + e^{f^2 \mathcal{K}_{f\sigma}} N^2 \right] \\
& \qquad \left.\left.\left. - f^2 e^{f^2 \mathcal{K}_{f\sigma}} \left[\frac{1}{1-N} \{ (N+1) f^2 + 8 - 10N \} + e^{f^2 \mathcal{K}_{f\sigma}} \frac{N^2}{1-N} \frac{1}{E} \right] \right] \right]\!\!\right]^{-1} .
\end{aligned}
\tag{7.1}
$$

Then the difference is evident. The solution for Z^2 must be primarily positive due to its square form and its positiveness should be analyzed. Under the same condition, a form of the solution for $<\chi_f>$ is also simplified as

$$
\begin{aligned}
<\chi_f>= (1-N)\Bigg[\!\!\Bigg[& g^2_{U(1)} \cdot \Bigg[2\,(1-N)\{f^2\xi + ((N+1)f^2 + 10 - 12N)E\} \\
& \quad -2f^2 e^{f^2\mathcal{K}_{f\sigma}} N + \frac{1}{2} f^2 e^{f^2\mathcal{K}_{f\sigma}} \left\{ 6 + \left(f^2\xi \frac{1}{E} + (N+1)f^2 - 2N \right) \frac{1}{1-N} \right\} \Bigg] \\
+ 2g^2_{SU(N+1)} \cdot \frac{1}{4} \Bigg[& \{-2\,(3-f^2)(1-N^2) + 8N(1-N)^2\}E - 4\Big[(1-N)\{(N+1)f^2 + 8 - 10N\}E + e^{f^2\mathcal{K}_{f\sigma}} N^2 \Big] \\
& + 16 e^{f^2\mathcal{K}_{f\sigma}} \left[f^2(1-N^2) + (2-f^2)\,\frac{1}{1-N} \right] - 2f^2 e^{f^2\mathcal{K}_{f\sigma}} - 4f^4 \frac{N^2}{(1-N)^2} e^{2f^2\mathcal{K}_{f\sigma}} \frac{1}{E} \\
& - f^2 e^{f^2\mathcal{K}_{f\sigma}} \left[\frac{1}{1-N}\{(N+1)f^2 + 8 - 10N\} + e^{f^2\mathcal{K}_{f\sigma}} \frac{N^2}{1-N} \frac{1}{E} \right] \Bigg] \Bigg]\!\!\Bigg] \\
\times \Bigg[\!\!\Bigg[& g^2_{U(1)} \cdot \left\{ 4\,(1-N)^2 E + f^2 e^{f^2\mathcal{K}_{f\sigma}} \right\} + 2g^2_{SU(N+1)} \cdot 2 \left[-\,(1-N)^2 E - \frac{1}{4} f^2 e^{f^2\mathcal{K}_{f\sigma}} \right] \Bigg]\!\!\Bigg]^{-1} .
\end{aligned}
\tag{7.2}
$$

To obtain the solutions for Z^2 and $<\chi_f>$, we have imposed a condition $(1-N|\mathbf{k}|^2)(1-Z^2) = |\mathbf{k}|^2 Z^2 <\chi_f> (N=5)$. What does this condition mean? It is an important and interesting problem to inquire upon the physical meaning of the condition, for example from the geometrical viewpoint. Through a method different from the one in [2], it is possible to determine simultaneously a solution of the quadratic equation for E because it is given in terms of only Z^2 and $<\chi_f>$. We here omit an explicit expression for the equation since it is very lengthy. Particularly in the case $|\mathbf{k}|^2 = 1$, the quadratic equation reduces to a simpler form.

In this paper, along the same strategy developed by van Holten et al. [2], we have embeded a coset coordinate in an anomaly-free spinor rep of $SO(2N+2)$ group and have given a corresponding Kähler potential and then a Killing potential for the anomaly-free $\frac{SO(2N+2)}{U(N+1)}$ model based on a positive chiral spinor rep. The theory is invariant under a SUSY transformation and the Killing potential is expressed in terms of the coset variables. To construct a consistent gauged version of the SUSY coset model, we must bring gauge fields into the model. Then the theory becomes no longer invariant under the transformation. To restore the SUSY, it is inevitable to introduce gauginos, auxiliary fields and a Fayet-Ilipoulos term. This makes the theory invariant under the SUSY transformation, i.e., chiral invariant and produces a new f-deformed reduced scalar potential. Using such mathematical manipulation we have thus constructed the anomaly-free $\frac{SO(2N+2)}{U(N+1)}$ SUSY σ-model and have investigated what are the new aspects which had not been seen previously in the SUSY σ-model on the Kähler coset space $\frac{SO(2N)}{U(N)}$.

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